CANONICAL METRICS ON THE MODULI SPACE OF RIEMANN SURFACES I

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Contents

1. Introductions

One of the main purpose of this paper is to compare those well-known canonical and complete metrics on the Teichmüller and the moduli spaces of Riemann surfaces. We use as bridge two new metrics, the Ricci metric and the perturbed Ricci metric. We will prove that these metrics are equivalent to those classical complete metrics. For this purpose we study in detail the asymptotic behaviors and the signs of the curvatures of these new metrics. In particular we prove that the perturbed Ricci metric is a complete Kähler metric with bounded negative holomorphic sectional curvature and bounded bisectional and Ricci curvature.

The study of the Teichmüller spaces and moduli spaces of Riemann surfaces has a long history. It has been intensively studied by many mathematicians in complex analysis, differential geometry, topology and algebraic geometry for the past 60 years. They have also appeared in theoretical physics such as string theory. The moduli space can be viewed as the quotient of the corresponding Teichmüller space by the modular group. There are several classical metrics on these spaces: the Weil-Petersson metric, the Teichmüller metric, the Kobayashi metric, the Bergman metric, the Caratheodory metric and the Kähler-Einstein metric. These metrics have been studied over the years and have found many important applications in various areas of mathematics. Each of these metrics has its own advantages and disadvantages in studying different problems.

The Weil-Petersson metric is a Kähler metric as first proved by Ahlfors, both of its holomorphic sectional curvature and Ricci curvature have negative upper bounds as conjectured by Royden and proved by Wolpert. These properties have found many applications by Wolpert, and they were also used in solving problems from algebraic geometry by combining with the Schwarz lemma of Yau ([5], [17]). But as first proved by Masur it is not a complete metric which prevents the understanding of some aspects of the geometry of the moduli spaces. Siu

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and Schumacher extended some results to higher dimensional cases. The works of Masur and Wolpert, Siu and Schumacher will play important roles in our study.

The Teichmüller metric, the Kobayashi metric and the Caratheodory metric are only Finsler metrics. They are very effective in studying the hyperbolic property of the moduli space. Royden proved that the Teichmüller metric is equal to the Kobayashi metric from which he deduced the important corollary that the isometry group of the Teichmüller space is exactly the modular group. Recently C. McMullen introduced a new complete Kähler metric on the moduli space by perturbing the Weil-Petersson metric [9]. By using this metric he was able to prove that the moduli space is Kähler hyperbolic, and also to derive several topological consequences. The McMullen metric has bounded geometry, but we lose control on the signs of its curvatures.

In the early 80s Cheng-Yau [2] and Mok-Yau [10] proved the existence of the Kähler-Einstein metrics on the Teichmüller space. Since the Kähler-Einstein metric is canonical, it also descends to a complete Kähler metric on the moduli space. More than 20 years ago Yau [18] conjectured the equivalence of the Kähler-Einstein metric to the Teichmüller metric. We will prove this conjecture in this paper. Since the McMullen metric is equivalent to the Teichmüller metric, so we have also proved the equivalence of the Kähler-Einstein metric and the McMullen metric.

The method of our proof is to study in detail another complete Kähler metric, the metric induced by the negative Ricci curvature of the Weil-Petersson metric which we call the Ricci metric. We first study its asymptotic behavior near the boundary of the moduli space, we prove that it is asymptotically equivalent to the Poincaré metric, and asymptotically its holomorphic sectional curvature has negative upper and lower bound in the degeneration directions. But its curvatures in the non-degeneration directions near the boundary and in the interior of the moduli space can not be controlled well. To solve this problem, we introduce another new complete Kähler metric which we call the perturbed Ricci metric, it is obtained by adding a multiple of the Weil-Petersson metric. We compute the holomorophic sectional curvature and the Ricci curvature of this new metric. We show that they are all bounded below and above, and the holomorphic sectional curvature has negative upper and lower bounds. By applying the Schwarz lemma of Yau we can prove the equivalence of this new metric to the Kähler-Einstein metric. The equivalence of the perturbed Ricci metric to the McMullen metric is proved by a careful estimate of the asymptotic behavior of these two metrics.

To state our main results in detail, let us introduce some definitions and notations. Here for convenience we will use the same notation for a Kähler metric and its Kähler form. First two metrics ω_{τ_1} and ω_{τ_2} are called equivalent, if they are quasi-isometric to each other in the sense that

$$C^{-1}\omega_{\tau_2} \leq \omega_{\tau_1} \leq C\omega_{\tau_2}$$

for some positive constant C. We will write this as $\omega_{\tau_1} \sim \omega_{\tau_2}$.

Our first result is the following asymptotic behavior of the Ricci metric near the boundary divisor of the moduli space. Let $\mathcal{T}_g$ denote the Teichmüller space and $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$ as a quotient of $\mathcal{T}_g$ by the modular group. Let $n = 3g-3$. Let ω_{WP} denote the Weil-Petersson metric and $\omega_\tau = -Ric(\omega_{WP})$ be the Ricci metric. It is easy to show that there is an asymptotic Poincaré metric on $\mathcal{M}_g$. See Section 4 for the construction.

Theorem 1.1. *The Ricci metric is equivalent to the asymptotic Poincaré metric.*

This theorem is proved in Section 4. Our second result is the following estimates of the holomorphic sectional curvature of the Ricci metric. Note our convention of the sign of the curvature may be different from some literature.

Theorem 1.2. *Let $X_0 \in \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ be a codimension m point and let $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates at X_0 where $t_1, \cdots, t_m$ correspond to the degeneration directions.*

Then the holomorphic sectional curvature of the Ricci metric is negative in the degeneration directions and is bounded in the non-degeneration directions. Precisely, there is a $\delta > 0$ such that if $|(t,s)| < \delta$, then

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0)) > 0 \quad \text{if} \quad i \le m$$

and

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \quad \text{if} \quad i \ge m+1.$$

Furthermore, on $\mathcal{M}_g$ the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.

This is Theorem 4.4 of Section 4 of this paper. One of the main purposes of our work was to find a natural complete metric whose holomorphic sectional curvature is negative. To do this, we introduce the perturbed Ricci metric. In Section 5 we will prove the following theorem:

Theorem 1.3. *For suitable choice of positive constant C, the perturbed Ricci metric*

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$$

is complete and its holomorphic sectional curvatures are negative and bounded from above and below by negative constants. Furthermore, the Ricci curvature of the perturbed Ricci metric is bounded from above and below.

Note that the perturbed Ricci metric is equivalent to the Ricci metric, since its asymptotic behavior is dominated by the Ricci metric. Now we denote the Kähler-Einstein metric of Cheng-Mok-Yau by ω_{KE} which is another complete Kähler metric on the moduli space. By applying the Schwarz lemma of Yau we derive our fourth result in Section 6:

Theorem 1.4. *We have the equivalence of the following three complete Kähler metrics on the moduli spaces of curves:*

$$\omega_{KE} \sim \omega_\tau \sim \omega_{\widetilde{\tau}}.$$

Our final result in this paper proved in Section 6 is the equivalence of the Ricci metric and the perturbed Ricci metric to the McMullen metric. Let us denote the McMullen metric by ω_M.

Theorem 1.5. *We have the equivalence of the following metrics: the McMullen metric, the Ricci metric and the perturbed Ricci metric:*

$$\omega_M \sim \omega_\tau \sim \omega_{\widetilde{\tau}}.$$

As a corollary we know that these metrics are also equivalent to the Teichmüller metric, the Kobayashi metric, and the Kähler-Einstein metric. This proved the conjecture of Yau [18]. In the second part of this work, we will study the Bergman metric and the Caratheodory metric. We believe that these two metrics are also equivalent to the above metrics. We will also study the goodness of the Ricci metric in the sense of Mumford, discuss the bounded geometry of the Kähler-Einstein metric and the perturbed Ricci metric, and study the stability of the tangent bundle of the moduli space of curves.

This paper is organized as follows. In Section 2 we set up some notations and introduce the Weil-Petersson metric and its curvatures. In Section 3 we introduce various operators needed for our computations, we compute and simplify the curvature of the Ricci metric by using these operators and their various special properties. This section consists of long and complicated computations. Section 4 consists of several subtle estimates of the Ricci metric and its curvatures near the boundary of the moduli space. In Section 5 we introduce the perturbed Ricci metric,

compute its curvature and study its asymptotic behavior near the boundary of the moduli space. These results are then used in Section 6 to prove the equivalence of the several well-known classical complete Kähler metrics as stated above. In the appendix we add some details of the computations for the convenience of the readers.

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2. The Weil-Petersson metric

The purpose of this section is to set up notations for our computations. We will introduce the Weil-Petersson metric and recall some of its basic properties. Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g where $g \geq 2$. $\mathcal{M}_g$ is a complex orbifold of dimension $3g-3$. Let $n = 3g-3$. Let $\mathfrak{X}$ be the total space and $\pi : \mathfrak{X} \to \mathcal{M}_g$ be the projection map. There is a natural metric, called the Weil-Petersson metric which is defined on the orbiford $\mathcal{M}_g$ as follows:

Let $s_1, \cdots, s_n$ be holomorphic local coordinates near a regular point $s \in \mathcal{M}_g$ and assume that z is a holomorphic local coordinate on the fiber $X_s = \pi^{-1}(s)$. For the holomorphic vector fields $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$, there are vector fileds $v_1, \cdots, v_n$ on $\mathfrak{X}$ such that

(1) $\pi_*(v_i) = \frac{\partial}{\partial s_i}$ for $i = 1, \cdots, n$;
(2) $\overline{\partial} v_i$ are harmonic TX_s-valued $(0,1)$ forms for $i = 1, \cdots, n$.

The vector fields $v_1, \cdots, v_n$ are called the harmonic lift of the vectors $\frac{\partial}{\partial s_1}, \cdots, \frac{\partial}{\partial s_n}$. The existence of such harmonic vector fields was pointed out by Siu [12]. In his work [11] Schumacher gave an explicit construction of such lift which we now describe.

Since $g \geq 2$, we can assume that each fiber is equipped with the Kähler-Einstein, or the Poincare metric, $\lambda = \frac{\sqrt{-1}}{2}\lambda(z,s) dz \wedge d\overline{z}$. The Kähler-Einstein condition gives the following equation:

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda. \tag{2.1}$$

For the rest of this paper we denote $\frac{\partial}{\partial s_i}$ by ∂_i and $\frac{\partial}{\partial z}$ by ∂_z. Let

$$a_i = -\lambda^{-1} \partial_i \partial_{\overline{z}} \log \lambda \tag{2.2}$$

and let

$$A_i = \partial_{\overline{z}} a_i. \tag{2.3}$$

Then we have the following

Lemma 2.1. *The harmonic horizontal lift of ∂_i is*

$$v_i = \partial_i + a_i \partial_z.$$

In particular

$$B_i = A_i \partial_z \otimes d\overline{z} \in H^1(X_s, T_{X_s})$$

is harmonic. Further more, the lift $\partial_i \mapsto B_i$ gives the Kodaira-Spencer map $T_s\mathcal{M}_g \to H^1(X_s, T_{X_s})$.

Now we have the well-known definition of the Weil-Petersson metric:

Definition 2.1. *The Weil-Petersson metric on $\mathcal{M}_g$ is defined to be*

$$h_{i\overline{j}}(s) = \int_{X_s} B_i \cdot \overline{B_j}\ dv = \int_{X_s} A_i \overline{A_j}\ dv, \tag{2.4}$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form on the fiber X_s.

It is known that the curvature tensor of the Weil-Petersson metric can be represented by

$$R_{i\bar{j}k\bar{l}} = \int_{X_s} \left\{ (B_i \cdot \overline{B}_j)(\Box + 1)^{-1}(B_k \cdot \overline{B}_l) + (B_i \cdot \overline{B}_l)(\Box + 1)^{-1}(B_k \cdot \overline{B}_j) \right\} dv,$$

where $\Box$ is the complex Laplacian defined by

$$\Box = -\lambda^{-1} \frac{\partial^2}{\partial z \partial \overline{z}}.$$

By the expression of the curvature operator, we know that the curvature operator is nonpositive. Furthermore, the Ricci curvature of the metric is negative.

However, the Weil-Petersson metric is incomplete. In [13] Trapani proved the negative Ricci curvature of the Weil-Petersson metric is a complete Kähler metric on the moduli space. We call this metric the Ricci metric. It is interesting to understand the curvature of the Ricci metric, at least asymptotically. To estimate it, we first derive an integral formula of its curvature.

3. Ricci metric and its curvature

In this section we establish an integral formula (3.30) of the curvature of the Ricci metric. The importance of this formula is that the functions being integrated only involve derivatives in the fiber direction which we are able to control. Thus we can use this formula to estimate the asymptotics of the curvature of the Ricci metric in next section.

The main tool we use is the harmonic lift of Siu and Schumacher described in the previous section. These lifts together with formula (3.2) enable us to transfer derivatives in the moduli direction into derivatives in the fiber direction.

We use the same notations as in the previous section. We first introduce several operators which will be used for the computations and simplifications of the curvatures of the Ricci metric.

Define an $(1,1)$ form on the total space $\mathfrak{X}$ by

$$g = \frac{\sqrt{-1}}{2} \partial\overline{\partial} \log \lambda = \frac{\sqrt{-1}}{2}(g_{i\bar{j}} ds_i \wedge d\overline{s}_j - \lambda a_i ds_i \wedge d\overline{z} - \lambda \overline{a}_i dz \wedge d\overline{s}_i + \lambda dz \wedge d\overline{z}).$$

The form g is not necessarily positive. Introduce

$$e_{i\bar{j}} = \frac{2}{\sqrt{-1}} g(v_i, \overline{v}_j) = g_{i\bar{j}} - \lambda a_i \overline{a_j}$$

be a global function. Let us write $f_{i\bar{j}} = A_i \overline{A_j}$. Schumacher proved the following result:

Lemma 3.1. *By using the same notations as above, we have*

$$(\Box + 1) e_{i\bar{j}} = f_{i\bar{j}}. \tag{3.1}$$

Since $e_{i\bar{j}}$ and $f_{i\bar{j}}$ are the building blocks of the Ricci metric, it is interesting to study its property under the action of the vector fields v_i's.

Lemma 3.2. *With the same notations as above, we have*

$$v_k(e_{i\bar{j}}) = v_i(e_{k\bar{j}}).$$

Proof. Since $dg = 0$, we have the following

$$\begin{aligned} 0 = dg(v_i, v_k, \overline{v}_j) &= v_i(e_{k\bar{j}}) - v_k(e_{i\bar{j}}) + \overline{v}_j g(v_i, v_k) \\ &\quad - g(v_i, [v_k, \overline{v}_j]) + g(v_k, [v_i, \overline{v}_j]) - g(\overline{v}_j, [v_i, v_k]). \end{aligned}$$

The Lie bracket of v_j with $\overline{v}_j$ or v_k are vector fields tangent to X_s, which are perpendicular to the horizontal vector fields v_i with respect to the form g. Thus the last three terms of the above equations are zero. On the other hand, $g(v_i, v_k) = 0$. The lemma thus follows from the above equation.

$\Box$

We also need to define the following operator

$$P:\ C^\infty(X_s)\to\Gamma(\Lambda^{1,0}(T^{0,1}X_s)),\ f\mapsto\partial_z(\lambda^{-1}\partial_z f).$$

The dual operator P^* can be written as follows

$$P^*:\ \Gamma(\Lambda^{0,1}(T^{1,0}X_s))\to C^\infty(X_s),\ B\mapsto\lambda^{-1}\partial_z(\lambda^{-1}\partial_z(\lambda B)).$$

The operator P is actually a composition of the Maass operators. We recall the definitions from [16]. Let X be a Riemann surface and let κ be its canonical bundle. For any integer p, let $S(p)$ be the space of smooth sections of $(\kappa\otimes\overline{\kappa}^{-1})^{\frac{p}{2}}$. Fix a conformal metric $ds^2=\rho^2(z)|dz|^2$.

Definition 3.1. *The Maass operators K_p and L_p are defined to be the metric derivatives $K_p:S(p)\to S(p+1)$ and $L_p:\ S(p)\to S(p-1)$ given by*

$$K_p(\sigma)=\rho^{p-1}\partial_z(\rho^{-p}\sigma)$$

and

$$L_p(\sigma)=\rho^{-p-1}\partial_{\overline{z}}(\rho^p\sigma)$$

where $\sigma\in S(p)$.

Clearly we have $P=K_1K_0$. Also each element $\sigma\in S(p)$ has a well-defined absolute value $|\sigma|$ which is independent of the choice of the local coordinate. We define the C^k norm of σ as in [16]:

Definition 3.2. *Let Q be an operator which is a composition of operators K_* and L_*. Denote by $|Q|$ the number of such factors. For any $\sigma\in S(p)$, define*

$$\|\sigma\|_0=\sup_X|\sigma|$$

and

$$\|\sigma\|_k=\sum_{|Q|\le k}\|Q\sigma\|_0.$$

We can also localize the norm on a subset of X. Let $\Omega\subset X$ be a domain. We can define

$$\|\sigma\|_{0,\Omega}=\sup_\Omega|\sigma|$$

and

$$\|\sigma\|_{k,\Omega}=\sum_{|Q|\le k}\|Q\sigma\|_{0,\Omega}.$$

Both of the above definitions depend on the choice of conformal metric on X. In the following, we always use the Kähler-Einstein metric on the surface unless otherwise stated.

Since the Weil-Petersson metric is defined by using the integral along the fibers, the following formula is very useful:

$$\partial_i\int_{X_s}\eta=\int_{X_s}L_{v_i}\eta \tag{3.2}$$

where η is a relative $(1,1)$ form on $\mathfrak{X}$.

The Lie derivative defined here is slightly different from the ordinary definition. Let φ_t be the one parameter group generated by the vector field v_i. Then φ_t can be viewed as a diffeomorphism between two fibers $X_s\to X_{s'}$. Then we define

$$L_{v_i}\eta=\lim_{t\to0}\frac{1}{t}(\varphi_t^*(\sigma)-\sigma)$$

for any one form σ. On the other hand, let ξ be a vector field on the fiber X_s. Then we define

$$L_{v_i}\xi = \lim_{t\to 0}\frac{1}{t}((\varphi_{-t})_*\xi - \xi).$$

We have the following

Proposition 3.1. *By using the above notations, we have*

$$L_{v_i}\sigma = i(v_i)d_1\sigma + d_1 i(v_i)\sigma,$$

where d_1 is the differential operator along the fiber, and

$$L_{v_i}\xi = [v_i, \xi].$$

In the following, we denote L_{v_i} by L_i.

Lemma 3.3. *By using the above notations, we have*
1. $L_i dv = 0$;
2. $L_{\bar{l}}(B_i) = -\overline{P}(e_{i\bar{l}}) - f_{i\bar{l}}\partial_{\bar{z}} \otimes d\bar{z} + f_{i\bar{l}}\partial_z \otimes dz$;
3. $L_k(\overline{B_j}) = -P(e_{k\bar{j}}) - f_{k\bar{j}}\partial_z \otimes dz + f_{k\bar{j}}\partial_{\bar{z}} \otimes d\bar{z}$;
4. $L_k(B_i) = (v_k(A_i) - A_i\partial_z a_k)\partial_z \otimes d\bar{z}$;
5. $L_{\bar{l}}(\overline{A_j}) = (\overline{v_l}(\overline{A_l}) - \overline{A_l}\partial_{\bar{z}}\overline{a_l})\partial_{\bar{z}} \otimes dz$.

Proof. The first formula was proved by Schumacher in [11]. To check the other formulas, we note that the third and fifth formulas follow from the second and fourth, which we will prove, by taking conjugation. We first have

$$\begin{aligned}\partial_z a_k &= \partial_z(-\lambda^{-1}\partial_k\partial_{\bar{z}}\log\lambda) = \lambda^{-2}\partial_z\lambda\partial_k\partial_{\bar{z}}\log\lambda - \lambda^{-1}\partial_z\partial_k\partial_{\bar{z}}\log\lambda\\ &= -\lambda^{-1}\partial_z\lambda a_k - \lambda^{-1}\partial_k\partial_z\partial_{\bar{z}}\log\lambda = -\lambda^{-1}\partial_z\lambda a_k - \lambda^{-1}\partial_k\lambda.\end{aligned}$$

We also have

$$\begin{aligned}\partial_{\bar{l}}a_i &= \partial_{\bar{l}}(-\lambda^{-1}\partial_i\partial_{\bar{z}}\log\lambda)) = \lambda^{-2}\partial_{\bar{l}}\lambda\partial_i\partial_{\bar{z}}\log\lambda - \lambda^{-1}\partial_{\bar{z}}\partial_i\partial_{\bar{l}}\log\lambda\\ &= -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}}g_{i\bar{l}} = -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}}(e_{i\bar{l}} + \lambda a_i\overline{a_l})\\ &= -\lambda^{-1}\partial_{\bar{l}}\lambda a_i - \lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}} - \lambda^{-1}\partial_{\bar{z}}\lambda a_i\overline{a_l} - A_i\overline{a_l} - a_i\partial_{\bar{z}}\overline{a_l}\\ &= -(\lambda^{-1}\partial_{\bar{l}}\lambda + \lambda^{-1}\partial_{\bar{z}}\lambda\overline{a_l} + \partial_{\bar{z}}\overline{a_l})a_i - \lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}} - A_i\overline{a_l}\\ &= -\lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}} - A_i\overline{a_l}.\end{aligned}$$

For the second formula we have

$$\begin{aligned}L_{\bar{l}}(B_i) &= \overline{v_l}(A_i)\partial_z \otimes d\bar{z} + A_i(-\partial_z\overline{a_l}\partial_{\bar{z}}) \otimes d\bar{z} + A_i\partial_z \otimes (\partial_z\overline{a_l}dz + \partial_{\bar{z}}\overline{a_l}d\bar{z})\\ &= (\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l})\partial_z \otimes d\bar{z} - f_{i\bar{l}}\partial_{\bar{z}} \otimes d\bar{z} + f_{i\bar{l}}\partial_z \otimes dz.\end{aligned}$$

So we only need to check that $\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l} = -\partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}})$. To prove this, we have

$$\begin{aligned}\overline{v_l}(A_i) + A_i\partial_{\bar{z}}\overline{a_l} &= \overline{a_l}\partial_{\bar{z}}A_i + \partial_{\bar{l}}A_i + A_i\partial_{\bar{z}}\overline{a_l} = \partial_{\bar{z}}(A_i\overline{a_l}) + \partial_{\bar{z}}\partial_{\bar{l}}a_i\\ &= \partial_{\bar{z}}(A_i\overline{a_l}) - \partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}}) - \partial_{\bar{z}}(A_i\overline{a_l}) = -\partial_{\bar{z}}(\lambda^{-1}\partial_{\bar{z}}e_{i\bar{l}}).\end{aligned}$$

This proved the second formula. For the fourth one, we have

$$L_k(B_i) = v_k(A_i)\partial_z \otimes d\bar{z} + A_i(-\partial_z a_k\partial_z) \otimes d\bar{z} = (v_k(A_i) - A_i\partial_z a_k)\partial_z \otimes d\bar{z}.$$

This finishes the proof.

□

An interesting and useful fact is that the Lie derivative of B_i in the direction of v_k is still harmonic. This result is true only for the moduli space of Riemann surfaces. In the general case of moduli space of Kähler -Einstein manifolds, we only have $\overline{\partial}^* L_k B_i = 0$.

Lemma 3.4. *$L_k(B_i) \in H^1(X_s, TX_s)$ is harmonic.*

Proof. From Lemma 3.3 we know that $L_k(B_i) = (v_k(A_i) - A_i\partial_z a_k)\partial_z \otimes d\overline{z} \in H^{0,1}(X_s, T_{X_s})$. So it is clear that $\overline{\partial}(L_k(B_i)) = 0$. To prove $\overline{\partial}^*(L_k(B_i)) = 0$ we only need to check that

$$\partial_z(\lambda(v_k(A_i) - A_i\partial_z a_k)) = 0.$$

From the computation in the above lemma, we have

$$\begin{aligned} v_k(A_i) - A_i\partial_z a_k =& \lambda a_i a_k - \partial_{\overline{z}}(\lambda^{-1}\partial_k\partial_i\partial_{\overline{z}}\log\lambda) \\ =& \lambda a_i a_k + \lambda^{-2}\partial_{\overline{z}}\lambda\partial_k\partial_i\partial_{\overline{z}}\log\lambda - \lambda^{-1}\partial_k\partial_i\partial_{\overline{z}}\partial_{\overline{z}}\log\lambda \end{aligned}$$

which implies

$$\begin{aligned} \partial_z(\lambda(v_k(A_i) - A_i\partial_z a_k)) =& \partial_z(\lambda^2 a_i a_k + \lambda^{-1}\partial_{\overline{z}}\lambda\partial_k\partial_i\partial_{\overline{z}}\log\lambda - \partial_k\partial_i\partial_{\overline{z}}\partial_{\overline{z}}\log\lambda) \\ =& \partial_z(\lambda^2 a_i a_k) + \partial_z(\lambda^{-1}\partial_{\overline{z}}\lambda)\partial_k\partial_i\partial_{\overline{z}}\log\lambda + \lambda^{-1}\partial_{\overline{z}}\lambda\partial_z(\partial_k\partial_i\partial_{\overline{z}}\log\lambda) \\ & - \partial_k\partial_i\partial_{\overline{z}}\partial_z\partial_{\overline{z}}\log\lambda \\ =& \partial_z(\lambda^2 a_i a_k) + \lambda\partial_k\partial_i\partial_{\overline{z}}\log\lambda + \lambda^{-1}\partial_{\overline{z}}\lambda\partial_k\partial_i\lambda - \partial_k\partial_i\partial_{\overline{z}}\lambda. \end{aligned} \tag{3.3}$$

Now we analyze the second term in (3.3). We have

$$\begin{aligned} \lambda\partial_k\partial_i\partial_{\overline{z}}\log\lambda =& \lambda\partial_k\partial_i\frac{\partial_{\overline{z}}\lambda}{\lambda} = \lambda\partial_k\frac{\lambda\partial_i\partial_{\overline{z}}\lambda - \partial_i\lambda\partial_{\overline{z}}\lambda}{\lambda^2} \\ =& \lambda\frac{\lambda^2(\partial_k\lambda\partial_i\partial_{\overline{z}}\lambda + \lambda\partial_k\partial_i\partial_{\overline{z}}\lambda - \partial_k\partial_i\lambda\partial_{\overline{z}}\lambda - \partial_i\lambda\partial_k\partial_{\overline{z}}\lambda)}{\lambda^4} \\ & - \lambda\frac{2\lambda\partial_k\lambda(\lambda\partial_i\partial_{\overline{z}}\lambda - \partial_i\lambda\partial_{\overline{z}}\lambda)}{\lambda^4} \\ =& -\lambda^{-1}\partial_k\lambda\partial_i\partial_{\overline{z}}\lambda + \partial_k\partial_i\partial_{\overline{z}}\lambda - \lambda^{-1}\partial_k\partial_i\lambda\partial_{\overline{z}}\lambda - \lambda^{-1}\partial_i\lambda\partial_k\partial_{\overline{z}}\lambda \\ & + 2\lambda^{-2}\partial_i\lambda\partial_k\lambda\partial_{\overline{z}}\lambda \\ =& -\partial_i\lambda(\lambda^{-1}\partial_k\partial_{\overline{z}}\lambda - \lambda^{-2}\partial_k\lambda\partial_{\overline{z}}\lambda) - \partial_k\lambda(\lambda^{-1}\partial_i\partial_{\overline{z}}\lambda - \lambda^{-2}\partial_i\lambda\partial_{\overline{z}}\lambda) \\ & + \partial_k\partial_i\partial_{\overline{z}}\lambda - \lambda^{-1}\partial_k\partial_i\lambda\partial_{\overline{z}}\lambda \\ =& -\partial_i\lambda\partial_k\partial_{\overline{z}}\log\lambda - \partial_k\lambda\partial_i\partial_{\overline{z}}\log\lambda + \partial_k\partial_i\partial_{\overline{z}}\lambda - \lambda^{-1}\partial_k\partial_i\lambda\partial_{\overline{z}}\lambda \\ =& \lambda\partial_i\lambda a_k + \lambda\partial_k\lambda a_i + \partial_k\partial_i\partial_{\overline{z}}\lambda - \lambda^{-1}\partial_k\partial_i\lambda\partial_{\overline{z}}\lambda. \end{aligned} \tag{3.4}$$

By combining (3.3) and (3.4) we have

$$\begin{aligned} \partial_z(\lambda(v_k(A_i) - A_i\partial_z a_k)) =& \partial_z(\lambda^2 a_i a_k) + \lambda\partial_i\lambda a_k + \lambda\partial_k\lambda a_i \\ =& 2\lambda\partial_z\lambda a_i a_k + \lambda^2\partial_z a_i a_k + \lambda^2 a_i\partial_z a_k + \lambda\partial_i\lambda a_k + \lambda\partial_k\lambda a_i \\ =& \lambda^2 a_k(\lambda^{-1}\partial_z\lambda a_i + \partial_z a_i + \lambda^{-1}\partial_i\lambda) \\ & + \lambda^2 a_i(\lambda^{-1}\partial_z\lambda a_k + \partial_z a_k + \lambda^{-1}\partial_k\lambda) \\ =& 0. \end{aligned}$$

This proves that $\overline{\partial}^*(L_k(B_i)) = 0$.

$\square$

The above lemma is very helpful in computing the curvature when we use normal coordinates of the Weil-Petersson metric. We have

Corollary 3.1. *Let $s_1, \cdots, s_n$ be normal coordinates at $s \in \mathcal{M}_g$ with respect to the Weil-Petersson metric. Then at s we have, for all i, k,*

$$L_k B_i = 0.$$

Proof. From Lemma 3.4 we know that L_kB_i is harmonic. Since $B_1, \cdots, B_n$ is a basis of $T_s\mathcal{M}_g$, we have

$$L_kB_i = h^{p\overline{q}}(\int_{X_s} L_kB_i \cdot \overline{B_q}\ dv)B_p = h^{p\overline{q}}\partial_k h_{i\overline{q}}B_p = 0.$$

□

The commutator of v_k and $\overline{v_l}$ will be used later. We give a formula here which is essentially due to Schumacher.

Lemma 3.5. $[\overline{v_l}, v_k] = -\lambda^{-1}\partial_{\overline{z}}e_{k\overline{l}}\partial_z + \lambda^{-1}\partial_z e_{k\overline{l}}\partial_{\overline{z}}$.

Proof. From a direct computation we have

$$[\overline{v_l}, v_k] = \overline{v_l}(a_k)\partial_z - v_k(\overline{a_l})\partial_{\overline{z}}.$$

By using Lemma 3.3 we have

$$\overline{v_l}(a_k) = \overline{a_l}\partial_{\overline{z}}a_k + \partial_{\overline{l}}a_k = -\lambda^{-1}\partial_{\overline{z}}e_{k\overline{l}}$$

and

$$v_k(\overline{a_l}) = a_k\partial_z\overline{a_l} + \partial_k\overline{a_l} = \lambda^{-1}\partial_z e_{k\overline{l}}.$$

These finish the proof.

□

Remark 3.1. In the rest of this paper, we will use the following notation for curvature: Let (M, g) be a Kähler manifold. Then the curvature tensor is given by

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} - g^{p\overline{q}}\frac{\partial g_{i\overline{q}}}{\partial z_k}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}. \tag{3.5}$$

In this situation, the Ricci curvature is given by

$$R_{i\overline{j}} = -g^{k\overline{l}}R_{i\overline{j}k\overline{l}}.$$

In [12] and [11], Siu and Schumacher proved the following curvature formula for the Weil-Petersson metric. This formula was also proved by Wolpert in [14]. Here we give a short proof here.

Theorem 3.1. *The curvature of Weil-Petersson metric is given by*

$$R_{i\overline{j}k\overline{l}} = \int_{X_s} (e_{i\overline{j}}f_{k\overline{l}} + e_{i\overline{l}}f_{k\overline{j}})\ dv. \tag{3.6}$$

Proof. We have

$$\begin{aligned} R_{i\overline{j}k\overline{l}} =& \partial_{\overline{l}}\partial_k h_{i\overline{j}} - h^{p\overline{q}}\partial_k h_{i\overline{q}}\partial_{\overline{l}}h_{p\overline{j}} \\ =& \partial_{\overline{l}}\int_{X_s} L_kB_i \cdot \overline{B_j}\ dv - h^{p\overline{q}}\int_{X_s} L_kB_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\overline{l}}\overline{B_j}\ dv \\ =& \int_{X_s} (L_{\overline{l}}L_kB_i \cdot \overline{B_j} + L_kB_i \cdot L_{\overline{l}}\overline{B_j})\ dv - h^{p\overline{q}}\int_{X_s} L_kB_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\overline{l}}\overline{B_j}\ dv. \end{aligned} \tag{3.7}$$

Since $B_1, \cdots, B_n$ is a basis of T_sM_g, we have

$$h^{p\overline{q}}\int_{X_s} L_kB_i \cdot \overline{B_q}\ dv \int_{X_s} B_p \cdot L_{\overline{l}}\overline{B_j}\ dv = \int_{X_s} L_kB_i \cdot L_{\overline{l}}\overline{B_j}\ dv.$$

By combining this formula with (3.7) we have

$$
\begin{aligned}
R_{i\bar{j}k\bar{l}} = & \int_{X_s} L_{\bar{l}} L_k B_i \cdot \overline{B_j}\ dv = \int_{X_s} L_k L_{\bar{l}} B_i \cdot \overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv \\
=& \partial_k \int_{X_s} L_{\bar{l}} B_i \cdot \overline{B_j}\ dv - \int_{X_s} L_{\bar{l}} B_i \cdot L_k \overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv \\
=& - \int_{X_s} L_{\bar{l}} B_i \cdot L_k \overline{B_j}\ dv + \int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv
\end{aligned} \tag{3.8}
$$

since $\int_{X_s} L_{\bar{l}} B_i \cdot \overline{B_j}\ dv = 0$. Now we compute $\int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv$. Let $\pi^1_1(L_{[\overline{v_l}, v_k]} B_i)$ be the projection of $L_{[\overline{v_l}, v_k]} B_i$ onto $H^{0,1}(X_s, T_{X_s})$ which gives the $\partial_z \otimes d\bar{z}$ part of $L_{[\overline{v_l}, v_k]} B_i$. Since B_i is harmonic, we know $\partial_z(\lambda A_i) = 0$ which implies $\partial_z A_i = -\lambda^{-1} \partial_z \lambda A_i$. By Lemma 3.5 we have

$$
\begin{aligned}
\pi^1_1(L_{[\overline{v_l}, v_k]} B_i) =& (-\lambda^{-1} \partial_{\bar{z}} e_{k\bar{l}} \partial_z A_i + A_i \partial_z(\lambda^{-1} \partial_{\bar{z}} e_{k\bar{l}}) + \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}})) \partial_z \otimes d\bar{z} \\
=& (\lambda^{-2} \partial_z \lambda A_i \partial_{\bar{z}} e_{k\bar{l}} - \lambda^{-2} \partial_z \lambda A_i \partial_{\bar{z}} e_{k\bar{l}} - A_i \Box e_{k\bar{l}} + \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}})) \partial_z \otimes d\bar{z} \\
=& (-A_i \Box e_{k\bar{l}} + \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}})) \partial_z \otimes d\bar{z}.
\end{aligned} \tag{3.9}
$$

This implies

$$
\begin{aligned}
\int_{X_s} L_{[\overline{v_l}, v_k]} B_i \cdot \overline{B_j}\ dv =& \int_{X_s} \pi^1_1(L_{[\overline{v_l}, v_k]} B_i) \cdot \overline{B_j}\ dv \\
=& \int_{X_s} (-A_i \Box e_{k\bar{l}} + \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}})) \overline{A_j}\ dv \\
=& - \int_{X_s} f_{i\bar{j}} \Box e_{k\bar{l}}\ dv + \int_{X_s} \partial_{\bar{z}}(\lambda^{-1} A_i \partial_z e_{k\bar{l}}) \overline{A_j}\ dv \\
=& - \int_{X_s} f_{i\bar{j}} \Box e_{k\bar{l}}\ dv - \int_{X_s} \lambda^{-2} A_i \partial_z e_{k\bar{l}} \partial_{\bar{z}}(\lambda \overline{A_j})\ dv \\
=& - \int_{X_s} f_{i\bar{j}} \Box e_{k\bar{l}}\ dv.
\end{aligned} \tag{3.10}
$$

To compute $\int_{X_s} L_{\bar{l}} B_i \cdot L_k \overline{B_j}\ dv$, by using Lemma 3.3 we obtain

$$
\begin{aligned}
&\int_{X_s} L_{\bar{l}} B_i \cdot L_k \overline{B_j}\ dv = \int_{X_s} (\partial_{\bar{z}}(\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}}) \partial_z(\lambda^{-1} \partial_z e_{k\bar{j}}) - 2 f_{k\bar{j}} f_{i\bar{l}})\ dv \\
=& \int_{X_s} (\lambda^{-2} \partial_z e_{k\bar{j}} \partial_z(\lambda \partial_{\bar{z}}(\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}}))\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& - \int_{X_s} (\lambda^{-2} \partial_z \lambda \partial_z e_{k\bar{j}} \partial_{\bar{z}}(\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}}) + \lambda^{-1} \partial_z e_{k\bar{j}} \partial_z \partial_{\bar{z}}(\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}}))\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& \int_{X_s} (\lambda^{-2} \partial_{\bar{z}} e_{i\bar{l}} \partial_{\bar{z}}(\lambda^{-1} \partial_z \lambda \partial_z e_{k\bar{j}}) + \lambda^{-1} \partial_z \partial_{\bar{z}} e_{k\bar{j}} \partial_z(\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}}))\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& \int_{X_s} (\lambda^{-2} \partial_{\bar{z}} e_{i\bar{l}} (\lambda \partial_z e_{k\bar{j}} - \partial_z \lambda \Box e_{k\bar{j}}) - \Box e_{k\bar{j}} (-\lambda^{-2} \partial_z \lambda \partial_{\bar{z}} e_{i\bar{l}} - \Box e_{i\bar{l}}))\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& \int_{X_s} (\lambda^{-1} \partial_{\bar{z}} e_{i\bar{l}} \partial_z e_{k\bar{j}}) + \Box e_{k\bar{j}} \Box e_{i\bar{l}})\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& \int_{X_s} (\Box e_{k\bar{j}} e_{i\bar{l}} + \Box e_{k\bar{j}} \Box e_{i\bar{l}})\ dv - 2 \int_{X_s} f_{k\bar{j}} f_{i\bar{l}}\ dv \\
=& \int_{X_s} (\Box e_{k\bar{j}} f_{i\bar{l}} - 2 f_{k\bar{j}} f_{i\bar{l}})\ dv = - \int_{X_s} (f_{k\bar{j}} f_{i\bar{l}} + e_{k\bar{j}} f_{i\bar{l}})\ dv.
\end{aligned} \tag{3.11}
$$

By combining (3.8), (3.10) and (3.11) with the identity $f_{k\bar{j}}f_{i\bar{l}} = A_i\overline{A_j}A_k\overline{A_l} = f_{i\bar{j}}f_{k\bar{l}}$, we have

$$
\begin{aligned}
R_{i\bar{j}k\bar{l}} &= \int_{X_s} (f_{k\bar{j}}f_{i\bar{l}} + e_{k\bar{j}}f_{i\bar{l}} - f_{i\bar{j}}\Box e_{k\bar{l}})\ dv = \int_{X_s} (f_{i\bar{j}}e_{k\bar{l}} + f_{i\bar{l}}e_{k\bar{j}})\ dv \\
&= \int_{X_s} (e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}})\ dv.
\end{aligned}
\tag{3.12}
$$

Here we have used the fact the $(\Box + 1)$ is a self-adjoint operator. This finished the proof. $\Box$

It is well-known that the Ricci curvature of the Weil-Petersson metric is negative which implies that the negative Ricci curvature of the Weil-Petersson metric defines a Kähler metric on the moduli space $\mathcal{M}_g$.

Definition 3.3. *The Ricci metric $\tau_{i\bar{j}}$ on the moduli space $\mathcal{M}_g$ is the negative Ricci curvature of the Weil-Petersson metric. That is*

$$
\tau_{i\bar{j}} = -R_{i\bar{j}} = h^{\alpha\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}. \tag{3.13}
$$

Now we define a new operator which acts on functions on the fibers.

Definition 3.4. *For each $1 \le k \le n$ and for any smooth function f on the fibers, we define the commutator operator ξ_k which acts on a function f by*

$$
\xi_k(f) = \overline{\partial}^*(i(B_k)\partial f) = -\lambda^{-1}\partial_z(A_k\partial_z f). \tag{3.14}
$$

The reason we call ξ_k the commutator operator is that ξ_k is the commutator of $(\Box + 1)$ and v_k and the following lemma.

Lemma 3.6. *As operators acting on functions, we have*
(1) $(\Box+1)v_k - v_k(\Box+1) = \Box v_k - v_k\Box = \xi_k$;
(2) $(\Box+1)\overline{v_l} - \overline{v_l}(\Box+1) = \Box\overline{v_l} - \overline{v_l}\Box = \overline{\xi_l}$;
(3) $\xi_k(f) = -A_k\partial_z(\lambda^{-1}\partial_z f) = A_kP(f) = -A_kK_1K_0(f)$.
Furthermore, we have

$$
(\Box + 1)v_k(e_{i\bar{j}}) = \xi_k(e_{i\bar{j}}) + \xi_i(e_{k\bar{j}}) + L_kB_i\cdot\overline{B_j}. \tag{3.15}
$$

Proof. To prove (1), we have

$$
\begin{aligned}
(\Box+1)v_k - v_k(\Box+1) =& \Box v_k + v_k - v_k\Box - v_k = \Box v_k - v_k\Box \\
=& -\lambda^{-1}\partial_z\partial_{\bar{z}}(a_k\partial_z + \partial_k) - (a_k\partial_z + \partial_k)(-\lambda^{-1}\partial_z\partial_{\bar{z}}) \\
=& -\lambda^{-1}\partial_z(A_k\partial_z + a_k\partial_z\partial_{\bar{z}} + \partial_k\partial_{\bar{z}}) \\
&+ a_k\partial_z(\lambda^{-1})\partial_z\partial_{\bar{z}} + \lambda^{-1}a_k\partial_z\partial_z\partial_{\bar{z}} + \partial_k(\lambda^{-1})\partial_z\partial_{\bar{z}} + \lambda^{-1}\partial_k\partial_z\partial_{\bar{z}} \\
=& -\lambda^{-1}\partial_z(A_k\partial_z) - \lambda^{-1}\partial_z a_k\partial_z\partial_{\bar{z}} - \lambda^{-1}a_k\partial_z\partial_z\partial_{\bar{z}} - \lambda^{-1}\partial_k\partial_z\partial_{\bar{z}} \\
&- \lambda^{-2}\partial_z\lambda a_k\partial_z\partial_{\bar{z}} + \lambda^{-1}a_k\partial_z\partial_z\partial_{\bar{z}} - \lambda^{-2}\partial_k\lambda\partial_z\partial_{\bar{z}} + \lambda^{-1}\partial_k\partial_z\partial_{\bar{z}} \\
=& \xi_k - \lambda^{-1}(\partial_z a_k + \lambda^{-1}\partial_z\lambda a_k + \lambda^{-1}\partial_k\lambda)\partial_z\partial_{\bar{z}} = \xi_k
\end{aligned}
$$

where we have used Lemma 3.3 in the last equality of the above formula. By taking conjugation we can prove (2) by using (1). To prove (3), we use the harmonicity of B_k. Since $\overline{\partial}^*B_k = 0$ we have $\partial_z(\lambda A_k) = 0$. So

$$
\xi_k(f) = -\lambda^{-1}\partial_z(A_k\partial_z f) = -\lambda^{-1}\partial_z(\lambda A_k\lambda^{-1}\partial_z f) = -\lambda^{-1}\lambda A_k\partial_z(\lambda^{-1}\partial_z f) = -A_k\partial_z(\lambda^{-1}\partial_z f).
$$

To prove the last part, by using part 1 of this lemma, we have

$$
\begin{aligned}
(\Box+1)v_k(e_{i\bar{j}}) =& v_k((\Box+1)(e_{i\bar{j}})) + \xi_k(e_{i\bar{j}}) = v_k(f_{i\bar{j}}) + \xi_k(e_{i\bar{j}}) \\
=& L_kB_i \cdot \overline{B_j} + B_i \cdot L_k\overline{B_j} + \xi_k(e_{i\bar{j}}) = L_kB_i \cdot \overline{B_j} - A_i\partial_z(\lambda^{-1}\partial_z e_{k\bar{j}}) + \xi_k(e_{i\bar{j}}) \\
=& L_kB_i \cdot \overline{B_j} + \xi_i(e_{k\bar{j}}) + \xi_k(e_{i\bar{j}}).
\end{aligned}
$$

This finishes the proof.

$\Box$

Remark 3.2. From Corollary 3.1 and the above lemma, when we use the normal coordinates on the moduli space, we have the clean formula $(\Box+1)v_k(e_{i\bar{j}}) = \xi_i(e_{k\bar{j}}) + \xi_k(e_{i\bar{j}})$.

The main result in this section is to prove the curvature formula of the Ricci metric. The terms produced here are very symmetric with respect to indices. For convenience, we introduce the symmetrization operator.

Definition 3.5. *Let U be any quantity which depends on indices $i, k, \alpha, \bar{j}, \bar{l}, \bar{\beta}$. The symmetrization operator σ_1 is defined by taking the summation of all orders of the triple (i, k, α). That is*

$$
\begin{aligned}
\sigma_1(U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta})) =& U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,\alpha,k,\bar{j},\bar{l},\bar{\beta}) + U(k,i,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(k,\alpha,i,\bar{j},\bar{l},\bar{\beta}) \\
&+ U(\alpha,i,k,\bar{j},\bar{l},\bar{\beta}) + U(\alpha,k,i,\bar{j},\bar{l},\bar{\beta}).
\end{aligned}
$$

Similarly, σ_2 is the symmetrization operator of $\bar{j}$ and $\bar{\beta}$ and $\widetilde{\sigma_1}$ is the symmetrization operator of $\bar{j}$, $\bar{l}$ and $\bar{\beta}$.

Now we are ready to compute the curvature of the Ricci metric. For the first order derivative we have

Theorem 3.2.

$$
\partial_k \tau_{i\bar{j}} = h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} (\xi_k(e_{i\bar{j}}) e_{\alpha\bar{\beta}}) \, dv \right\} + \tau_{p\bar{j}} \Gamma^p_{ik} \tag{3.16}
$$

where Γ^p_{ik} is the Christoffell symbol of the Weil-Petersson metric.

Proof. From Lemma 3.1 we know that $(\Box+1)e_{i\bar{j}} = f_{i\bar{j}}$. By using Lemma 3.6 and Theorem 3.1 we have

$$
\begin{aligned}
\partial_k R_{i\bar{j}\alpha\bar{\beta}} =& \partial_k \int_{X_s} (e_{i\bar{j}} f_{\alpha\bar{\beta}} + e_{i\bar{\beta}} f_{\alpha\bar{j}}) \, dv \\
=& \int_{X_s} (v_k(e_{i\bar{j}}) f_{\alpha\bar{\beta}} + e_{i\bar{j}} v_k(f_{\alpha\bar{\beta}}) + v_k(e_{i\bar{\beta}}) f_{\alpha\bar{j}} + e_{i\bar{\beta}} v_k(f_{\alpha\bar{j}})) \, dv \\
=& \int_{X_s} ((\Box+1)v_k(e_{i\bar{j}}) e_{\alpha\bar{\beta}} + e_{i\bar{j}} v_k(f_{\alpha\bar{\beta}}) + (\Box+1)v_k(e_{i\bar{\beta}}) e_{\alpha\bar{j}} + e_{i\bar{\beta}} v_k(f_{\alpha\bar{j}})) \, dv \\
=& \int_{X_s} (v_k(f_{i\bar{j}}) e_{\alpha\bar{\beta}} + e_{i\bar{j}} v_k(f_{\alpha\bar{\beta}}) + v_k(f_{i\bar{\beta}}) e_{\alpha\bar{j}} + e_{i\bar{\beta}} v_k(f_{\alpha\bar{j}})) \, dv \\
&+ \int_{X_s} (\xi_k(e_{i\bar{j}}) e_{\alpha\bar{\beta}} + \xi_k(e_{i\bar{\beta}}) e_{\alpha\bar{j}}) \, dv \\
=& \int_{X_s} ((L_kB_i \cdot \overline{B_j}) e_{\alpha\bar{\beta}} + (L_kB_\alpha \cdot \overline{B_\beta}) e_{i\bar{j}} + (L_kB_i \cdot \overline{B_\beta}) e_{\alpha\bar{j}} + (L_kB_\alpha \cdot \overline{B_j}) e_{i\bar{\beta}}) \, dv \\
&+ \int_{X_s} ((B_i \cdot L_k\overline{B_j}) e_{\alpha\bar{\beta}} + (B_\alpha \cdot L_k\overline{B_\beta}) e_{i\bar{j}} + (B_i \cdot L_k\overline{B_\beta}) e_{\alpha\bar{j}} + (B_\alpha \cdot L_k\overline{B_j}) e_{i\bar{\beta}}) \, dv \\
&+ \int_{X_s} (\xi_k(e_{i\bar{j}}) e_{\alpha\bar{\beta}} + \xi_k(e_{i\bar{\beta}}) e_{\alpha\bar{j}}) \, dv.
\end{aligned} \tag{3.17}
$$

Now we simplify the right hand side of (3.17). Since $B_1, \cdots, B_n$ is a basis of T_sM_g, we know that the first line of the right hand side of (3.17) is

$$
\begin{aligned}
&\int_{X_s} ((L_kB_i\cdot\overline{B_j})e_{\alpha\overline{\beta}} + (L_kB_\alpha\cdot\overline{B_\beta})e_{i\overline{j}} + (L_kB_i\cdot\overline{B_\beta})e_{\alpha\overline{j}} + (L_kB_\alpha\cdot\overline{B_j})e_{i\overline{\beta}})\ dv\\
=&\int_{X_s} (L_kB_i\cdot(\overline{B_j}e_{\alpha\overline{\beta}} + \overline{B_\beta}e_{\alpha\overline{j}}) + L_kB_\alpha\cdot(\overline{B_j}e_{i\overline{\beta}} + \overline{B_\beta}e_{i\overline{j}}))\ dv\\
=&h^{p\overline{q}}\int_{X_s}(L_kB_i\cdot\overline{B_q})\ dv\int_{X_s}(B_p\cdot(\overline{B_j}e_{\alpha\overline{\beta}} + \overline{B_\beta}e_{\alpha\overline{j}})\ dv\\
&+h^{p\overline{q}}\int_{X_s}(L_kB_\alpha\cdot\overline{B_q})\ dv\int_{X_s}(B_p\cdot(\overline{B_j}e_{i\overline{\beta}} + \overline{B_\beta}e_{i\overline{j}})\ dv\\
=&h^{p\overline{q}}\partial_kh_{i\overline{q}}R_{p\overline{j}\alpha\overline{\beta}} + h^{p\overline{q}}\partial_kh_{\alpha\overline{q}}R_{i\overline{j}p\overline{\beta}} = \Gamma^p_{ik}R_{p\overline{j}\alpha\overline{\beta}} + \Gamma^p_{\alpha k}R_{i\overline{j}p\overline{\beta}}.
\end{aligned}
\tag{3.18}
$$

We deal with the second line of the right hand side of (3.17) by using Lemma 3.3 and Lemma 3.6 to get

$$
B_i\cdot L_k\overline{B_j} = -A_i\partial_z(\lambda^{-1}\partial_ze_{k\overline{j}}) = \xi_i(e_{k\overline{j}}).
\tag{3.19}
$$

This implies

$$
\begin{aligned}
&\int_{X_s}((B_i\cdot L_k\overline{B_j})e_{\alpha\overline{\beta}} + (B_\alpha\cdot L_k\overline{B_\beta})e_{i\overline{j}} + (B_i\cdot L_k\overline{B_\beta})e_{\alpha\overline{j}} + (B_\alpha\cdot L_k\overline{B_j})e_{i\overline{\beta}})\ dv\\
=&\int_{X_s}(\xi_i(e_{k\overline{j}})e_{\alpha\overline{\beta}} + \xi_\alpha(e_{k\overline{\beta}})e_{i\overline{j}} + \xi_i(e_{k\overline{\beta}})e_{\alpha\overline{j}} + \xi_\alpha(e_{k\overline{j}})e_{i\overline{\beta}})\ dv.
\end{aligned}
\tag{3.20}
$$

We also have

$$
\partial_k\tau_{i\overline{j}} = h^{\alpha\overline{\beta}}\partial_kR_{i\overline{j}\alpha\overline{\beta}} + \partial_kh^{\alpha\overline{\beta}}R_{i\overline{j}\alpha\overline{\beta}} = h^{\alpha\overline{\beta}}(\partial_kR_{i\overline{j}\alpha\overline{\beta}} - R_{i\overline{j}p\overline{\beta}}\Gamma^p_{k\alpha}).
\tag{3.21}
$$

By combining (3.17), (3.18), (3.20) and (3.21), together with the fact that ξ_i is a real symmetric operator and the definition of $\tau_{i\overline{j}}$, we have proved this theorem.

□

To compute the second order derivative, we need to compute the commutator of ξ_k and $\overline{v_l}$. We have

Lemma 3.7. *For any smooth function $f\in C^\infty(X_s)$,*

$$
\overline{v_l}(\xi_kf) - \xi_k(\overline{v_l}f) = \overline{P}(e_{k\overline{l}})P(f) - 2f_{k\overline{l}}\Box f + \lambda^{-1}\partial_zf_{k\overline{l}}\partial_{\overline{z}}f.
\tag{3.22}
$$

Proof. We will fix local holomorphic coordinates and compute locally. First we know that the commutator of $\overline{v_l}$ and ∂_z is

$$
\overline{v_l}\partial_z - \partial_z\overline{v_l} = -\partial_z\overline{a_l}\partial_{\overline{z}} = -\overline{A_l}\partial_{\overline{z}}.
\tag{3.23}
$$

Similarly, the commutator of $\overline{v_l}$ and $\lambda^{-1}\partial_z$ is

$$
\overline{v_l}(\lambda^{-1}\partial_z) - \lambda^{-1}\partial_z\overline{v_l} = \overline{v_l}(\lambda^{-1})\partial_z + \lambda^{-1}(\overline{v_l}\partial_z - \partial_z\overline{v_l}) = \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z - \lambda^{-1}\overline{A_l}\partial_{\overline{z}}.
\tag{3.24}
$$

The above two formulae imply

$$
\begin{aligned}
\overline{v_l}P - P\overline{v_l} =& -\overline{v_l}(\partial_z(\lambda^{-1}\partial_z)) + \partial_z(\lambda^{-1}\partial_z)\overline{v_l}\\
=&(\overline{A_l}\partial_{\overline{z}} - \partial_z\overline{v_l})(\lambda^{-1}\partial_z) + \partial_z(\overline{v_l}(\lambda^{-1}\partial_z) - \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z + \lambda^{-1}\overline{A_l}\partial_{\overline{z}})\\
=&\overline{A_l}\partial_{\overline{z}}(\lambda^{-1}\partial_z) - \partial_z(\lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z) + \partial_z(\lambda^{-1}\overline{A_l}\partial_{\overline{z}})\\
=&-\lambda^{-2}\partial_{\overline{z}}\lambda\overline{A_l}\partial_z + \lambda^{-1}\overline{A_l}\partial_z\partial_{\overline{z}} + \lambda^{-2}\partial_z\lambda\partial_{\overline{z}}\overline{a_l}\partial_z - \lambda^{-1}\partial_{\overline{z}}\overline{A_l}\partial_z - \lambda^{-1}\partial_{\overline{z}}\overline{a_l}\partial_z\partial_z\\
&-\lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\partial_z\overline{A_l}\partial_{\overline{z}} + \lambda^{-1}\overline{A_l}\partial_z\partial_{\overline{z}}.
\end{aligned}
\tag{3.25}
$$

By using the harmonicity, we have $\partial_{\bar z}(\lambda\overline{A_l})=0$ which implies $\partial_{\bar z}\overline{A_l}=-\lambda^{-1}\partial_{\bar z}\lambda\overline{A_l}$. By plugging this into formula (3.25) we have

$$\begin{aligned}\overline{v_l}P-P\overline{v_l}=&-2\overline{A_l}\Box+\lambda^{-2}\partial_z\lambda\partial_{\bar z}\overline{a_l}\partial_z-\lambda^{-1}\partial_{\bar z}\overline{a_l}\partial_z\partial_z-\lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\bar z}+\lambda^{-1}\partial_z\overline{A_l}\partial_{\bar z}\\=&-2\overline{A_l}\Box+\partial_{\bar z}\overline{a_l}P-\lambda^{-2}\partial_z\lambda\overline{A_l}\partial_{\bar z}+\lambda^{-1}\partial_z\overline{A_l}\partial_{\bar z}.\end{aligned}\tag{3.26}$$

Now, since $\xi_k=A_kP$, we have

$$\begin{aligned}\overline{v_l}(\xi_kf)-\xi_k(\overline{v_l}f)=&\overline{v_l}(A_k)P(f)+A_k(\overline{v_l}P(f)-P\overline{v_l}(f))\\=&(\overline{v_l}(A_k)+A_k\partial_{\bar z}\overline{a_l})P(f)-2f_{k\bar l}\Box f-\lambda^{-2}\partial_z\lambda A_k\overline{A_l}\partial_{\bar z}+\lambda^{-1}A_k\partial_z\overline{A_l}\partial_{\bar z}.\end{aligned}\tag{3.27}$$

From the proof of lemma 3.3 we know $\overline{v_l}(A_k)+A_k\partial_{\bar z}\overline{a_l}=\overline{P}(e_{k\bar l})$. By using the harmonicity we have $-\lambda^{-1}\partial_z\lambda A_k=\partial_zA_k$. So from (3.27) we have

$$\begin{aligned}\overline{v_l}(\xi_kf)-\xi_k(\overline{v_l}f)=&\overline{P}(e_{k\bar l})P(f)-2f_{k\bar l}\Box f+\lambda^{-1}\partial_zA_k\overline{A_l}\partial_{\bar z}f+\lambda^{-1}A_k\partial_z\overline{A_l}\partial_{\bar z}f\\=&\overline{P}(e_{k\bar l})P(f)-2f_{k\bar l}\Box f+\lambda^{-1}\partial_zf_{k\bar l}\partial_{\bar z}f.\end{aligned}\tag{3.28}$$

This finishes the proof.

□

From the above lemma, it is convenient to define the commutator of ξ_k and $\overline{v_l}$ as an operator.

Definition 3.6. *For each k,l, we define the operator $Q_{k\bar l}$ which acts on a function to produce another function by*

$$Q_{k\bar l}(f)=\overline{P}(e_{k\bar l})P(f)-2f_{k\bar l}\Box f+\lambda^{-1}\partial_zf_{k\bar l}\partial_{\bar z}f.\tag{3.29}$$

Now we are ready to compute the curvature tensor of the Ricci metric. The formula consists of four types of terms.

Theorem 3.3. *Let $s_1,\cdots,s_n$ be local holomorphic coordinates at $s\in M_g$. Then at s, we have*

$$\begin{aligned}\widetilde{R}_{i\bar jk\bar l}=&h^{\alpha\bar\beta}\left\{\sigma_1\sigma_2\int_{X_s}\left\{(\Box+1)^{-1}(\xi_k(e_{i\bar j}))\overline{\xi}_l(e_{\alpha\bar\beta})+(\Box+1)^{-1}(\xi_k(e_{i\bar j}))\overline{\xi}_\beta(e_{\alpha\bar l})\right\}\ dv\right\}\\&+h^{\alpha\bar\beta}\left\{\sigma_1\int_{X_s}Q_{k\bar l}(e_{i\bar j})e_{\alpha\bar\beta}\ dv\right\}\\&-\tau^{p\bar q}h^{\alpha\bar\beta}h^{\gamma\bar\delta}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar q})e_{\alpha\bar\beta}\ dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar j})e_{\gamma\bar\delta})\ dv\right\}\\&+\tau_{p\bar j}h^{p\bar q}R_{i\bar qk\bar l}.\end{aligned}\tag{3.30}$$

Proof. By Lemma 3.4 we know that L_kB_i is harmonic. Since $B_1,\cdots,B_n$ is a basis of harmonic Beltrami differentials, from the proof of Theorem 3.1 we have

$$L_kB_i=\Gamma^s_{ik}B_s.\tag{3.31}$$

We first compute $\partial_{\bar{l}} \int_{X_s} \xi_k(e_{i\bar{j}}) e_{\alpha\bar{\beta}}\, dv$. By Lemma 3.6 and Lemma 3.7 we have

$$
\begin{aligned}
\partial_{\bar{l}} \int_{X_s} \xi_k(e_{i\bar{j}}) e_{\alpha\overline{\beta}}\, dv =& \int_{X_s} (\overline{v}_l(\xi_k(e_{i\bar{j}}))e_{\alpha\overline{\beta}} + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}))\, dv \\
=& \int_{X_s} (\xi_k(\overline{v}_l(e_{i\bar{j}}))e_{\alpha\overline{\beta}} + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}) + Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}})\, dv \\
=& \int_{X_s} (\xi_k(e_{\alpha\overline{\beta}})\overline{v}_l(e_{i\bar{j}}) + \xi_k(e_{i\bar{j}})\overline{v}_l(e_{\alpha\overline{\beta}}) + Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}})\, dv \\
=& \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\Box+1)(\overline{v}_l(e_{i\bar{j}}))\, dv \\
&+ \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))(\Box+1)(\overline{v}_l(e_{\alpha\overline{\beta}}))\, dv + \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\, dv \\
=& \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\xi}_l(e_{i\bar{j}}) + \overline{v}_l(f_{i\bar{j}}))\, dv \\
&+ \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))(\overline{\xi}_l(e_{\alpha\overline{\beta}}) + \overline{v}_l(f_{\alpha\overline{\beta}}))\, dv \\
&+ \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\, dv \\
=& \int_{X_s} ((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))\overline{\xi}_l(e_{i\bar{j}}) + (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}))\, dv \\
&+ \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\xi}_j(e_{i\bar{l}}) + A_i \cdot L_{\bar{l}}\overline{A_j})\, dv \\
&+ \int_{X_s} (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))(\overline{\xi}_\beta(e_{\alpha\bar{l}}) + A_\alpha \cdot L_{\bar{l}}\overline{A_\beta})\, dv \\
&+ \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\, dv.
\end{aligned}
\tag{3.32}
$$

Now by using (3.31) we have

$$
\begin{aligned}
&\int_{X_s} ((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(A_i \cdot L_{\bar{l}}\overline{A_j}) + (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))(A_\alpha \cdot L_{\bar{l}}\overline{A_\beta}))\, dv \\
=& \int_{X_s} ((\Box+1)^{-1}(\xi_k(e_{\alpha\overline{\beta}}))(\overline{\Gamma^t_{jl}} A_i \cdot \overline{A_t}) + (\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))(\overline{\Gamma^t_{\beta l}} A_\alpha \cdot \overline{A_t}))\, dv \\
=& \overline{\Gamma^t_{jl}} \int_{X_s} \xi_k(e_{\alpha\overline{\beta}})(\Box+1)^{-1}(A_i \cdot \overline{A_t})\, dv + \overline{\Gamma^t_{\beta l}} \int_{X_s} \xi_k(e_{i\bar{j}})(\Box+1)^{-1}(A_\alpha \cdot \overline{A_t})\, dv \\
=& \overline{\Gamma^t_{jl}} \int_{X_s} \xi_k(e_{\alpha\overline{\beta}})e_{i\bar{t}}\, dv + \overline{\Gamma^t_{\beta l}} \int_{X_s} \xi_k(e_{i\bar{j}})e_{\alpha\bar{t}}\, dv.
\end{aligned}
\tag{3.33}
$$

By combining (3.32) and (3.33) we have

$$
\begin{aligned}
\partial_{\bar l}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv=&\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\bar j}))(\overline{\xi}_l(e_{\alpha\bar\beta})+\overline{\xi}_\beta(e_{\alpha\bar l}))\,dv\\
&+\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{\alpha\bar\beta}))(\overline{\xi}_l(e_{i\bar j})+\overline{\xi}_j(e_{i\bar l}))\,dv\\
&+\overline{\Gamma^t_{jl}}\int_{X_s}\xi_k(e_{\alpha\bar\beta})e_{i\bar t}\,dv+\overline{\Gamma^t_{\beta l}}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar t}\,dv\\
&+\int_{X_s}Q_{k\bar l}(e_{i\bar j})e_{\alpha\bar\beta}\,dv.
\end{aligned}\tag{3.34}
$$

We also have

$$
\begin{aligned}
\partial_{\bar l}\Gamma^p_{ik}=&\partial_{\bar l}(h^{p\bar q}\partial_k h_{i\bar q})=-h^{p\bar\beta}h^{\alpha\bar q}\partial_{\bar l}h_{\alpha\bar\beta}\partial_k h_{i\bar q}+h^{p\bar q}\partial_{\bar l}\partial_k h_{i\bar q}\\
=&h^{p\bar q}(\partial_{\bar l}\partial_k h_{i\bar q}-h^{\alpha\bar\beta}\partial_{\bar l}h_{\alpha\bar q}\partial_k h_{i\bar\beta})=h^{p\bar q}R_{i\bar q k\bar l}.
\end{aligned}\tag{3.35}
$$

From Theorem 3.2, formula (3.34) and (3.35) we derive

$$
\begin{aligned}
\partial_{\bar l}\partial_k\tau_{i\bar j}=&(\partial_{\bar l}h^{\alpha\bar\beta})\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}+h^{\alpha\bar\beta}\left\{\sigma_1\partial_{\bar l}\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}\\
&+h^{\gamma\bar\delta}\left\{\widetilde\sigma_1\int_{X_s}\overline{\xi}_l(e_{p\bar j})e_{\gamma\bar\delta}\,dv\right\}\Gamma^p_{ik}+\tau_{p\bar q}\Gamma^p_{ik}\overline{\Gamma^q_{jl}}+\tau_{p\bar j}h^{p\bar q}R_{i\bar q k\bar l}\\
=&-h^{\alpha\bar t}\overline{\Gamma^\beta_{lt}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}\\
&+h^{\alpha\bar\beta}\left\{\sigma_1\sigma_2\int_{X_s}(\Box+1)^{-1}(\xi_k(e_{i\bar j}))(\overline{\xi}_l(e_{\alpha\bar\beta})+\overline{\xi}_\beta(e_{\alpha\bar l}))\,dv\right\}\\
&+h^{\alpha\bar\beta}\left\{\sigma_1\int_{X_s}Q_{k\bar l}(e_{i\bar j})e_{\alpha\bar\beta}\,dv\right\}+h^{\alpha\bar\beta}\overline{\Gamma^t_{jl}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar t})e_{\alpha\bar\beta}\,dv\right\}\\
&+h^{\alpha\bar\beta}\overline{\Gamma^t_{\beta l}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\bar j})e_{\alpha\bar t}\,dv\right\}+h^{\gamma\bar\delta}\left\{\widetilde\sigma_1\int_{X_s}\overline{\xi}_l(e_{p\bar j})e_{\gamma\bar\delta}\,dv\right\}\Gamma^p_{ik}\\
&+\tau_{p\bar q}\Gamma^p_{ik}\overline{\Gamma^q_{jl}}+\tau_{p\bar j}h^{p\bar q}R_{i\bar q k\bar l}.
\end{aligned}\tag{3.36}
$$

Now from the above formula, by using Theorem 3.2 we can easily check the formula (3.30).

□

The curvature formula of the Ricci metric would be simpler if we have used the normal coordinates. However, when we estimate the asymptotic behavior of the curvature, it is hard to describe the normal coordinates near the boundary points. Thus we will use this general formula directly in our computations. The estimates are quite subtle.

4. The asymptotics of the Ricci metric and its curvatures

From formula (3.6) we can easily see the sign of the curvature of the Weil-Petersson metric directly. However, the sign of the curvature of the Ricci metric cannot be derived from formula (3.30). In this section, we estimate the asymptotics of the Ricci metric and its curvatures. We first describe the local pinching coordinates near the boundary of the moduli space due to the plumbing construction of Wolpert. Then we use Masur's construction of the holomorphic quadratic differentials to estimate the harmonic Beltrami differentials. Finally, we construct $\widetilde e_{i\bar j}$ which is an approximation of $e_{i\bar j}$. By doing this we avoid the estimates of the Green function of $\Box+1$ on the Riemann surfaces.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification [3]. Each point $y \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ corresponds to a stable nodal surface X_y. A point $p \in X_y$ is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v) \mid uv = 0,\ |u|,|v| < 1\} \subset \mathbb{C}^2$.

We first recall the rs-coordinate on a Riemann surface defined by Wolpert in [16]. There are two cases: the puncture case and the short geodesic case. For the puncture case, we have a nodal surface X and a node $p \in X$. Let a, b be two punctures which are glued together to form p.

Definition 4.1. *A local coordinate chart (U, u) near a is called rs-coordinate if $u(a) = 0$ where u maps U to the punctured disc $0 < |u| < c$ with $c > 0$, and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.*

For the short geodesic case, we have a closed surface X, a closed geodesic $\gamma \subset X$ with length $l < c_*$ where c_* is the collar constant.

Definition 4.2. *A local coordinate chart (U, z) is called rs-coordinate at γ if $\gamma \subset U$ where z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$, and the Kähler-Einstein metric on X can be written as $\frac{1}{2}(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|})^2|dz|^2$.*

Remark 4.1. We put the factor $\frac{1}{2}$ in the above two definitions to normalize metrics such that (2.1) hold.

By Keen's collar theorem [4], we have the following lemma:

Lemma 4.1. *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

(1) *z maps Ω to the annulus $\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c$ for $c > 0$;*
(2) *the Kähler-Einstein metric on X restricted to Ω is given by*

$$(\frac{1}{2}u^2 r^{-2}\csc^2\tau)|dz|^2 \tag{4.1}$$

where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;

(3) *the geodesic γ is given by the equation $|z| = e^{-\frac{\pi^2}{l}}$.*

We call such a collar Ω a genuine collar.

We notice that the constant c in the above lemma has a lower bound such that the area of Ω is bounded from below. Also, the coordinate z in the above lemma is rs-coordinate. In the following, we will keep using the above notations u, r and τ.

Now we describe the local manifold cover of $\overline{\mathcal{M}}_g$ near the boundary. We take the construction of Wolpert [16]. Let $X_{0,0}$ be a nodal surface corresponding to a codimension m boundary point. $X_{0,0}$ have m nodes $p_1, \cdots, p_m$. $X_0 = X_{0,0} \setminus \{p_1, \cdots, p_m\}$ is a union of punctured Riemann surfaces. Fix the rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \cdots, m$ such that all the U_i and V_i are mutually disjoint. Now pick an open set $U_0 \subset X_0$ such that the intersection of each connected component of X_0 and U_0 is a nonempty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. Now pick Beltrami differentials $\nu_{m+1}, \cdots, \nu_n$ which are supported in U_0 and span the tangent space at X_0 of the deformation space of X_0. For $s = (s_{m+1}, \cdots, s_n)$, let $\nu(s) = \sum_{i=m+1}^n s_i\nu_i$. We assume $|s| = (\sum|s_i|^2)^{\frac{1}{2}}$ small enough such that $|\nu(s)| < 1$. The nodal surface $X_{0,s}$ is obtained by solving the Beltrami equation $\overline{\partial}w = \nu(s)\partial w$. Since $\nu(s)$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates on $X_{0,s}$. Note that they are

no longer rs-coordinates. By the theory of Alhfors and Bers [1] and Wolpert [16] we can assume that there are constants $\delta, c > 0$ such that when $|s| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,s}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$. Now we assume $t = (t_1, \cdots, t_m)$ has small norm. We do the plumbing construction on $X_{0,s}$ to obtain $X_{t,s}$. We remove from $X_{0,s}$ the discs $0 < |\eta_i| \le \frac{|t_i|}{c}$ and $0 < |\zeta_i| \le \frac{|t_i|}{c}$ for each $i = 1, \cdots, m$, and identify $\frac{|t_i|}{c} < |\eta_i| < c$ with $\frac{|t_i|}{c} < |\zeta_i| < c$ by the rule $\eta_i \zeta_i = t_i$. This defines the surface $X_{t,s}$. The tuple $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar defined by $\frac{|t_i|}{c} < |\eta_i| < c$ the plumbing collar.

Remark 4.2. From the estimate of Wolpert [15], [16] on the length of short geodesic, we have $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log |t_i|}$.

We also need the following version of the Schauder estimate proved by Wolpert [16].

Theorem 4.1. *Let X be a closed Riemann surface equipped with the unique Kähler-Einstein metric. Let f and g be smooth functions on X such that $(\Box + 1)g = f$. Then for any integer $k \ge 0$, there is a constant c_k such that $\|g\|_{k+1} \le c_k \|f\|_k$ where the norm is defined by* (3.2).

Now we estimate the asymptotics of the Ricci metric in the pinching coordinates. We will use the following notations. Let $(t, s) = (t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ be the pinching coordinates near $X_{0,0}$. For $|(t, s)| < \delta$, let Ω_c^j be the j-th genuine collar on $X_{t,s}$ which contains a short geodesic γ_j with length l_j. Let $u_j = \frac{l_j}{2\pi}$, $u_0 = \sum_{j=1}^m u_j + \sum_{j=m+1}^n |s_j|$, $r_j = |z_j|$ and $\tau_j = u_j \log r_j$ where z_j is the properly normalized rs-coordinate on Ω_c^j such that

$$\Omega_c^j = \{z_j \mid c^{-1} e^{-\frac{2\pi^2}{l_j}} < |z_j| < c\}.$$

From the above argument, we know that the Kähler-Einstein metric λ on $X_{t,s}$ restrict to the collar Ω_c^j is given by

$$\lambda = \frac{1}{2} u_j^2 r_j^{-2} \csc^2 \tau_j. \tag{4.2}$$

For convenience, we let $\Omega_c = \cup_{j=1}^m \Omega_c^j$ and $R_c = X_{t,s} \setminus \Omega_c$. In the following, we may change the constant c finitely many times, clearly this will not affect the estimates.

To estimate the curvature of the Ricci metric, the first step is to find all the harmonic Beltrami differentials $B_1, \cdots, B_n$ which correspond to the tangent vectors $\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial s_n}$. In [8], Masur constructed $3g-3$ regular holomorphic quadratic differentials $\psi_1, \cdots, \psi_n$ on the plumbing collars by using the plumbing coordinate η_j. These quadratic differentials correspond to the cotangent vectors $dt_1, \cdots, ds_n$.

However, it is more convenient to estimate the curvature if we use the rs-coordinate on $X_{t,s}$ since we have the accurate form of the Kähler-Einstein metric λ in this coordinate. In [13], Trapani used the graft metric constructed by Wolpert [16] to estimate the difference between the plumbing coordinate and rs-coordinate and gave the holomorphic quadratic differentials constructed by Masur in the rs-coordinate. We collect Trapani's results (Lemma 6.2-6.5, [13]) in the following theorem:

Theorem 4.2. *Let (t, s) be the pinching coordinates on $\overline{\mathcal{M}}_g$ near $X_{0,0}$ which corresponds to a codimension m boundary point of $\overline{\mathcal{M}}_g$. Then there exist constants $M, \delta > 0$ and $1 > c > 0$ such that if $|(t, s)| < \delta$, then the j-th plumbing collar on $X_{t,s}$ contains the genuine collar Ω_c^j. Furthermore, one can choose rs-coordinate z_j on the collar Ω_c^j properly such that the holomorphic quadratic differentials $\psi_1, \cdots, \psi_n$ corresponding to the cotangent vectors $dt_1, \cdots, ds_n$ have the form $\psi_i = \varphi_i(z_j) dz_j^2$ on the genuine collar Ω_c^j for $1 \le j \le m$, where*

(1) $\varphi_i(z_j) = \frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $i \geq m+1$*;*
(2) $\varphi_i(z_j) = (-\frac{t_j}{\pi})\frac{1}{z_j^2}(q_j(z_j) + \beta_j)$ *if* $i = j$*;*
(3) $\varphi_i(z_j) = (-\frac{t_i}{\pi})\frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $1 \leq i \leq m$ *and* $i \neq j$*.*

Here β_i^j *and* β_j *are functions of* (t,s)*,* q_i^j *and* q_j *are functions of* (t,s,z_j) *given by*

$$q_i^j(z_j) = \sum_{k<0} \alpha_{ik}^j(t,s) t_j^{-k} z_j^k + \sum_{k>0} \alpha_{ik}^j(t,s) z_j^k$$

and

$$q_j(z_j) = \sum_{k<0} \alpha_{jk}(t,s) t_j^{-k} z_j^k + \sum_{k>0} \alpha_{jk}(t,s) z_j^k$$

such that
(1) $\sum_{k<0} |\alpha_{ik}^j| c^{-k} \leq M$ *and* $\sum_{k>0} |\alpha_{ik}^j| c^k \leq M$ *if* $i \neq j$*;*
(2) $\sum_{k<0} |\alpha_{jk}| c^{-k} \leq M$ *and* $\sum_{k>0} |\alpha_{jk}| c^k \leq M$*;*
(3) $|\beta_i^j| = O(|t_j|^{\frac{1}{2}-\epsilon})$ *with* $\epsilon < \frac{1}{2}$ *if* $i \neq j$*;*
(4) $|\beta_j| = (1 + O(u_0))$*.*

An immediate consequence of the above theorem is the following refined version of Masur's estimates of the Weil-Petersson metric. In the following, we will fix (t,s) with small norm and let $X = X_{t,s}$.

Corollary 4.1. *Let* (t,s) *be the pinching coordinates. Then*
(1) $h^{i\bar{i}} = 2u_i^{-3}|t_i|^2(1 + O(u_0))$ *and* $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1 + O(u_0))$ *for* $1 \leq i \leq m$*;*
(2) $h^{i\bar{j}} = O(|t_i t_j|)$ *and* $h_{i\bar{j}} = O(\frac{u_i^3 u_j^3}{|t_i t_j|})$*, if* $1 \leq i,j \leq m$ *and* $i \neq j$*;*
(3) $h^{i\bar{j}} = O(1)$ *and* $h_{i\bar{j}} = O(1)$*, if* $m+1 \leq i,j \leq n$*;*
(4) $h^{i\bar{j}} = O(|t_i|)$ *and* $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \leq m < j$ *or* $j \leq m < i$*.*

Proof. We need the following simple calculus results:

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} \frac{1}{r_j} \sin^2 \tau_j \, dr_j = u_j^{-1}(\frac{\pi}{2} + O(u_j)). \tag{4.3}$$

For any $k \geq 1$,

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} r_j^{k-1} \sin^2 \tau_j \, dr_j = O(u_j^2) c^k \tag{4.4}$$

and for $k \leq -1$,

$$\int_{c^{-1}e^{-\frac{2\pi^2}{l_j}}}^{c} r_j^{k-1} \sin^2 \tau_j \, dr_j = O(u_j^2) c^{-k} \left(e^{-\frac{2\pi^2}{l_j}} \right)^k. \tag{4.5}$$

On the collar Ω_c^j, the metric λ is given by (4.2). $h^{i\bar{j}}$ is given by the formula

$$h^{i\bar{j}} = \int_X \psi_i \overline{\psi_j} \lambda^{-2} dv.$$

By using the above calculus facts, we can compute the above integral on the collars. The bound on R_c was calculated in [8]. A simple computation shows that the first part of all of the above claims hold. The second parts of these claims can be obtained by inverting the matrix $(h^{i\bar{j}})$ together with Masur's result on the nondegenerate extension of the submatrix $(h^{i\bar{j}})_{i,j>m}$.This finishes the proof.

□

Now we are ready to compute the harmonic Beltrami differentials $B_i = A_i \partial_z \otimes d\bar{z}$.

Lemma 4.2. *For c small, on the genuine collar Ω_c^j, the coefficient functions A_i of the harmonic Beltrami differentials have the form:*

(1) $A_i = \frac{z_j}{\overline{z_j}} \sin^2 \tau_j \big(\overline{p_i^j(z_j)} + \overline{b_i^j} \big)$ *if* $i \neq j$;
(2) $A_j = \frac{z_j}{\overline{z_j}} \sin^2 \tau_j \big(\overline{p_j(z_j)} + \overline{b_j} \big)$

where

(1) $p_i^j(z_j) = \sum_{k \leq -1} a_{ik}^j \rho_j^{-k} z_j^k + \sum_{k \geq 1} a_{ik}^j z_j^k$ *if* $i \neq j$;
(2) $p_j(z_j) = \sum_{k \leq -1} a_{jk} \rho_j^{-k} z_j^k + \sum_{k \geq 1} a_{jk} z_j^k$.

In the above expressions, $\rho_j = e^{-\frac{2\pi^2}{l_j}}$ *and the coefficients satisfy the following conditions:*

(1) $\sum_{k \leq -1} |a_{ik}^j| c^{-k} = O(u_j^{-2})$ *and* $\sum_{k \geq 1} |a_{ik}^j| c^k = O(u_j^{-2})$ *if* $i \geq m+1$;
(2) $\sum_{k \leq -1} |a_{ik}^j| c^{-k} = O(u_j^{-2}) O\big(\frac{u_i^3}{|t_i|}\big)$ *and* $\sum_{k \geq 1} |a_{ik}^j| c^k = O(u_j^{-2}) O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \leq m$ *and* $i \neq j$;
(3) $\sum_{k \leq -1} |a_{jk}| c^{-k} = O\big(\frac{u_j}{|t_j|}\big)$ *and* $\sum_{k \geq 1} |a_{jk}| c^k = O\big(\frac{u_j}{|t_j|}\big)$;
(4) $|b_i^j| = O(u_j)$ *if* $i \geq m+1$;
(5) $|b_i^j| = O(u_j) O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \leq m$ *and* $i \neq j$;
(6) $b_j = -\frac{u_j}{\pi \overline{t_j}} (1 + O(u_0))$.

Proof. The duality between the harmonic Beltrami differentials and the holomorphic quadratic differentials is given by

$$B_i = \lambda^{-1} \sum_{l=1}^{n} h_{i\bar{l}} \overline{\psi_l} \tag{4.6}$$

which implies $A_i = \lambda^{-1} \sum_{l=1}^{n} h_{i\bar{l}} \overline{\varphi_l}$. Now by Wolpert's estimate on the length of the short geodesic γ_j in [16] we have $l_j = -\frac{2\pi^2}{\log |t_j|}(1 + O(u_j))$. This implies there is a constant $0 < \mu < 1$ such that $\mu |t_j| < \rho_j < \mu^{-1} |t_j|$. The lemma follows from equation (4.6) by replacing c by μc, a simple computation together with Theorem 4.2 and Corollary 4.1.

□

To estimate the curvature of the Ricci metric, we need to estimate the asymptotics of the Ricci metric by using Theorem 3.1. So we need the following estimates on the norms of the harmonic Beltrami differentials.

Lemma 4.3. *Let $\| \cdot \|_k$ be the norm as defined in Definition 3.2. We have*

(1) $\|A_i\|_{0, \Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|A_i\|_{0, X \setminus \Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i \leq m$;
(2) $\|A_i\|_0 = O(1)$, *if* $i \geq m+1$;
(3) $\|f_{i\bar{i}}\|_{0, \Omega_c^i} = O\big(\frac{u_i^2}{|t_i|^2}\big)$ *and* $\|f_{i\bar{i}}\|_{0, X \setminus \Omega_c^i} = O\big(\frac{u_i^6}{|t_i|^2}\big)$, *if* $i \leq m$;
(4) $\|f_{i\bar{j}}\|_0 = O(1)$, *if* $i, j \geq m+1$;
(5) $\|f_{i\bar{j}}\|_{0, \Omega_c^i} = O\big(\frac{u_i u_j^3}{|t_i t_j|}\big)$ *and* $\|f_{i\bar{j}}\|_{0, \Omega_c^j} = O\big(\frac{u_i^3 u_j}{|t_i t_j|}\big)$ *and* $\|f_{i\bar{j}}\|_{0, X \setminus (\Omega_c^i \cup \Omega_c^j)} = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$ *if* $i, j \leq m$ *and* $i \neq j$;
(6) $\|f_{i\bar{j}}\|_{0, \Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|f_{i\bar{j}}\|_{0, X \setminus \Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i \leq m$ *and* $j \geq m+1$;
(7) $|f_{i\bar{j}}|_{L^1} = O(1)$, *if* $i, j \geq m+1$;
(8) $|f_{i\bar{j}}|_{L^1} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i \leq m$ *and* $j \geq m+1$;
(9) $|f_{i\bar{j}}|_{L^1} = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$, *if* $i, j \leq m$ *and* $i \neq j$.

Proof. We choose c small enough such that for each $1 \leq j \leq m$,

$$\tan(u_j \log c) < -10 u_j$$

when $|(t,s)|<\delta$. A simple computation shows that, when $1\le p\le 10$, on the collar Ω_c^j we have

$$|r_j^k \sin^p \tau_j| \le c^k |\log c|^p u_j^p$$

if $k\ge 1$, and

$$|r_j^k \sin^p \tau_j| \le c^{-k} |\log c|^p \rho_j^k u_j^p$$

if $k\le -1$.

To prove the first claim, note that on Ω_c^i we have

$$\begin{aligned}|A_i| =& \left|\frac{z_i}{\overline{z_i}}\right| |\sin^2\tau_i(\overline{p_i}+\overline{b_i})| \le \sum_{k\le -1} |a_{ik}|\rho_i^{-k} r_i^k \sin^2\tau_i + \sum_{k\ge 1}|a_{ik}| r_i^k \sin^2\tau_i + |b_j| \\ \le&(\log c)^2 u_i^2\Big(\sum_{k\le -1}|a_{ik}|c^{-k}+\sum_{k\ge 1}|a_{ik}|c^k\Big)+|b_j| \\ =&O(u_i^2)O\big(\frac{u_i}{|t_i|}\big)+O(u_i^2)O\big(\frac{u_i}{|t_i|}\big)+O\big(\frac{u_i}{|t_i|}\big)=O\big(\frac{u_i}{|t_i|}\big).\end{aligned}$$

Similarly, on Ω_c^j with $j\ne i$, we have $|A_i| = O\big(\frac{u_i^3}{|t_i|}\big)$. Also, on R_c we have $|A_i| = O\big(\frac{u_i^3}{|t_i|}\big)$ by the work of Masur [8], equation (4.6) together with Theorem 4.2 and Corollary 4.1. This finishes the proof of the first claim.

The second claim can be proved in a similar way. Claim (3)-(6) follow from the first and second claims by using the fact that $f_{i\bar{j}} = A_i\overline{A_j}$. Claim (7) follows from claim (4) and the fact that the area of X is a fixed positive constant using the Gauss-Bonnet theorem.

Now we prove claim (9). On Ω_c^i, by using a similar estimate as above, we have

$$\begin{aligned}|f_{i\bar{j}}| =&|\sin^4\tau_i(\overline{p_i}+\overline{b_i})(p_j^i+b_j^i)| \le |\sin^4\tau_i\overline{p_i}p_j^i| + |\sin^4\tau_i\overline{b_i}p_j^i| + |\sin^4\tau_i\overline{p_i}b_j^i| + |\sin^4\tau_i\overline{b_i}b_j^i| \\ \le&O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)+|\sin^4\tau_i\overline{b_i}b_j^i| = O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)+O\big(\frac{u_i^2u_j^3}{|t_it_j|}\big)\sin^4\tau_i.\end{aligned}$$

So

$$|f_{i\bar{j}}|_{L^1(\Omega_c^i)} \le \int_{\Omega_c^i}\Big(O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)+O\big(\frac{u_i^2u_j^3}{|t_it_j|}\big)\sin^4\tau_i\Big)dv = O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big).$$

Similarly, $|f_{i\bar{j}}|_{L^1(\Omega_c^j)} \le O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)$. The estimate $|f_{i\bar{j}}|_{L^1(X\setminus(\Omega_c^i\cup\Omega_c^j))} = O\big(\frac{u_i^3u_j^3}{|t_it_j|}\big)$ follows from claim (5). This proves claim (9). Similarly we can prove claim (8).

□

In the following, we will denote the operator $(\square+1)^{-1}$ by T. We then have the following estimates about L^2 norms:

Lemma 4.4. *Let $f\in C^\infty(X,\mathbb{C})$. Then we have*

$$\int_X |Tf|^2\ dv \le \int_X Tf\cdot\overline{f}\ dv \le \int_X |f|^2\ dv. \tag{4.7}$$

Proof. This lemma is a simple application of the spectral decomposition of the operator $(\square+1)$ and the fact that all eigenvalues of this operator are greater than or equal to 1. One can also prove it directly by using integration by part.

□

To estimate the Ricci metric, we also need to estimate the functions $e_{i\bar{j}}$. We localize these functions on the collars by constructing the following approximation functions.

Pick a positive constant $c_1<c$ and define the cut-off function $\eta\in C^\infty(\mathbb{R},[0,1])$ by

$$\begin{cases}\eta(x)=1, & x\le\log c_1;\\ \eta(x)=0, & x\ge \log c;\\ 0<\eta(x)<1, & \log c_1<x<\log c.\end{cases} \tag{4.8}$$

It is clear that the derivatives of η are bounded by constants which only depend on c and c_1. Let $\widetilde{e_{i\bar{j}}}(z)$ be the function on X defined in the following way where z is taken to be z_i on the collar Ω_c^i:

(1) if $i \le m$ and $j \ge m+1$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i, & z \in \Omega_{c_1}^i; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log r_i), & z \in \Omega_c^i \text{ and } c_1 < r_i < c; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log\rho_i - \log r_i), & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i; \\ 0, & z \in X \setminus \Omega_c^i; \end{cases}$$

(2) if $i, j \le m$ and $i \ne j$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i, & z \in \Omega_{c_1}^i; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log r_i), & z \in \Omega_c^i \text{ and } c_1 < r_i < c; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i} b_j^i)\eta(\log\rho_i - \log r_i), & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i; \\ \frac{1}{2}\sin^2\tau_j \overline{b_i^j} b_j, & z \in \Omega_{c_1}^j; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i^j} b_j)\eta(\log r_j), & z \in \Omega_c^j \text{ and } c_1 < r_j < c; \\ (\frac{1}{2}\sin^2\tau_i \overline{b_i^j} b_j)\eta(\log\rho_j - \log r_j), & z \in \Omega_c^j \text{ and } c^{-1}\rho_j < r_j < c_1^{-1}\rho_j; \\ 0, & z \in X \setminus (\Omega_c^i \cup \Omega_c^j); \end{cases}$$

(3) if $i \le m$, then

$$\widetilde{e_{i\bar{i}}}(z) = \begin{cases} \frac{1}{2}\sin^2\tau_i |b_i|^2, & z \in \Omega_{c_1}^i; \\ (\frac{1}{2}\sin^2\tau_i |b_i|^2)\eta(\log r_i), & z \in \Omega_c^i \text{ and } c_1 < r_i < c; \\ (\frac{1}{2}\sin^2\tau_i |b_i|^2)\eta(\log\rho_i - \log r_i), & z \in \Omega_c^i \text{ and } c^{-1}\rho_i < r_i < c_1^{-1}\rho_i; \\ 0, & z \in X \setminus \Omega_c^i. \end{cases}$$

Also, let $\widetilde{f_{i\bar{j}}} = (\Box + 1)\widetilde{e_{i\bar{j}}}$. It is clear that the supports of these approximation functions are contained in the corresponding collars. We have the following estimates:

Lemma 4.5. *Let $\widetilde{e_{i\bar{j}}}$ be the functions constructed above. Then*

(1) $e_{i\bar{i}} = \widetilde{e_{i\bar{i}}} + O\big(\frac{u_i^4}{|t_i|^2}\big)$, *if $i \le m$;*
(2) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$, *if $i, j \le m$ and $i \ne j$;*
(3) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3}{|t_i|}\big)$, *if $i \le m$ and $j \ge m+1$;*
(4) $\|e_{i\bar{j}}\|_0 = O(1)$, *if $i, j \ge m+1$.*

Proof. The last claim follows from the maximum principle and Lemma 4.3. To prove the first claim, we note that the maximum principle implies

$$\|e_{i\bar{i}} - \widetilde{e_{i\bar{i}}}\|_0 \le \|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}\|_0.$$

Now we compute the right hand side of the above inequality. Since $\widetilde{f_{i\bar{i}}}\,|_{X\setminus\Omega_c^i} = 0$, by Lemma 4.3 we know that $\|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^6}{|t_i|^2}\big)$. On $\Omega_{c_1}^i$ we have

$$|f_{i\bar{i}} - \widetilde{f_{i\bar{i}}}| \le |\sin^4\tau_i \overline{p_i} b_i| + |\sin^4\tau_i \overline{b_i} p_i| + |\sin^4\tau_i \overline{p_i} p_i| = O\big(\frac{u_i^6}{|t_i|^2}\big)$$

which implies $\|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}\|_{0,\Omega_{c_1}^i}=O\big(\frac{u_i^6}{|t_i|^2}\big)$. On $\Omega_c^i\setminus\Omega_{c_1}^i$ with $c_1\le r_i\le c$, we have

$$\begin{aligned}|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}|\le&(1-\eta)|b_i|^2\sin^4\tau_i+|\sin^4\tau_i\overline{p_i}b_i|+|\sin^4\tau_i\overline{b_i}p_i|+|\sin^4\tau_i\overline{p_i}p_i|\\&+\frac{|b_i|^2u_i^{-2}|\eta''|}{4}\sin^4\tau_i+\frac{|b_i|^2u_i^{-1}|\eta'|}{2}\sin^2\tau_i|\sin 2\tau_i|\\=&O\big(\frac{u_i^4}{|t_i|^2}\big).\end{aligned}$$

Similarly, on $\Omega_c^i\setminus\Omega_{c_1}^i$ with $c^{-1}\rho_i\le r_i\le c_1^{-1}\rho_i$, we have $|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}|\le O\big(\frac{u_i^4}{|t_i|^2}\big)$. By combining the above estimate, we have $\|f_{i\bar{i}}-\widetilde{f_{i\bar{i}}}\|_0=O\big(\frac{u_i^4}{|t_i|^2}\big)$ which implies the first claim. The second and the third claims can be proved in a similar way.

□

As a corollary we prove the following estimates which are more refined than those of Trapani's on the Ricci metric [13]. The precise constants of the leading terms will be used later to compute the curvature of the Ricci metric.

Corollary 4.2. *Let (t,s) be the pinching coordinates. Then we have*

(1) $\tau_{i\bar{i}}=\frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *and* $\tau^{i\bar{i}}=\frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1+O(u_0))$, *if* $i\le m$;
(2) $\tau_{i\bar{j}}=O\bigg(\frac{u_i^2u_j^2}{|t_it_j|}(u_i+u_j)\bigg)$ *and* $\tau^{i\bar{j}}=O(|t_it_j|)$, *if* $i,j\le m$ *and* $i\ne j$;
(3) $\tau_{i\bar{j}}=O\big(\frac{u_i^2}{|t_i|}\big)$ *and* $\tau^{i\bar{j}}=O(|t_i|)$, *if* $i\le m$ *and* $j\ge m+1$;
(4) $\tau_{i\bar{j}}=O(1)$, *if* $i,j\ge m+1$.

Remark 4.3. The second part of the above corollary can be made sharper. However, it will not be useful for our later estimates.

Proof. The second part of the corollary is obtained by inverting the matrix $(\tau_{i\bar{j}})$ in the first part together with the fact that the matrix $(h_{i\bar{j}})_{i,j\ge m+1}$ is nondegenerate which was proved by Masur and the fact that the matrix $(\tau_{i\bar{j}})_{i,j\ge m+1}$ is bounded from below by a constant multiple of the matrix $(h_{i\bar{j}})_{i,j\ge m+1}$ which was proved by Wolpert.

Now we prove the first part. In the following, we use C_0 to denote all universal constants which may change. Recall that

$$\tau_{i\bar{j}}=h^{\alpha\bar{\beta}}R_{i\bar{j}\alpha\bar{\beta}}. \tag{4.9}$$

To prove the last claim, let $i,j\ge m+1$. We first notice that if $\alpha\ne\beta$ or $\alpha=\beta\ge m+1$, then $|h^{\alpha\bar{\beta}}|\|A_\alpha\|_0\|A_\beta\|_0=O(1)$ by Lemma 4.3 and Corollary 4.1. In this case, we have

$$\begin{aligned}|R_{i\bar{j}\alpha\bar{\beta}}|\le&\left|\int_X e_{i\bar{j}}f_{\alpha\bar{\beta}}\,dv\right|+\left|\int_X e_{i\bar{\beta}}f_{\alpha\bar{j}}\,dv\right|\le C_0(\|e_{i\bar{j}}\|_0\|f_{\alpha\bar{\beta}}\|_0+\|e_{i\bar{\beta}}\|_0\|f_{\alpha\bar{j}}\|_0)\\\le&C_0(\|f_{i\bar{j}}\|_0\|f_{\alpha\bar{\beta}}\|_0+\|f_{i\bar{\beta}}\|_0\|f_{\alpha\bar{j}}\|_0)=O(1)\|A_\alpha\|_0\|A_\beta\|_0\end{aligned}$$

which implies $|h^{\alpha\overline{\beta}}R_{i\overline{j}\alpha\overline{\beta}}|=O(1)$. If $\alpha=\beta\leq m$ we have

$$\begin{aligned}|R_{i\overline{j}\alpha\overline{\alpha}}|\leq&\left|\int_X e_{i\overline{j}}f_{\alpha\overline{\alpha}}\ dv\right|+\left|\int_X e_{i\overline{\alpha}}f_{\alpha\overline{j}}\ dv\right|\leq(\|e_{i\overline{j}}\|_0|f_{\alpha\overline{\alpha}}|_{L^1}+\left(\int_X|e_{i\overline{\alpha}}|^2\ dv\int_X|f_{\alpha\overline{j}}|^2\ dv\right)^{\frac{1}{2}}\\ \leq&O(1)O\big(\frac{u_i^3}{|t_i|^2}\big)+\left(\int_X|f_{i\overline{\alpha}}|^2\ dv\int_X|f_{\alpha\overline{j}}|^2\ dv\right)^{\frac{1}{2}}\\ =&O\big(\frac{u_i^3}{|t_i|^2}\big)+\left(\int_X f_{i\overline{i}}f_{\alpha\overline{\alpha}}\ dv\int_X f_{\alpha\overline{\alpha}}f_{j\overline{j}}\ dv\right)^{\frac{1}{2}}\leq O\big(\frac{u_i^3}{|t_i|^2}\big)+\|A_i\|_0\|A_j\|_0|f_{\alpha\overline{\alpha}}|_{L^1}=O\big(\frac{u_i^3}{|t_i|^2}\big)\end{aligned}$$

which implies $|h^{\alpha\overline{\alpha}}R_{i\overline{j}\alpha\overline{\alpha}}|=O(1)$. So we have proved that last claim.

To prove the third claim, let $i\leq m$ and $j\geq m+1$. If $\alpha\neq\beta$ or $\alpha=\beta\geq m+1$ in formula (4.9), by using integration by part we have

$$\begin{aligned}|R_{i\overline{j}\alpha\overline{\beta}}|\leq&\left|\int_X f_{i\overline{j}}e_{\alpha\overline{\beta}}\ dv\right|+\left|\int_X f_{i\overline{\beta}}e_{\alpha\overline{j}}\ dv\right|\leq C_0(\|e_{\alpha\overline{\beta}}\|_0|f_{i\overline{j}}|_{L^1}+\|e_{\alpha\overline{j}}\|_0|f_{i\overline{\beta}}|_{L^1})\\ \leq&C_0(\|f_{\alpha\overline{\beta}}\|_0|f_{i\overline{j}}|_{L^1}+\|f_{\alpha\overline{j}}\|_0|f_{i\overline{\beta}}|_{L^1})=O\big(\frac{u_i^3}{|t_i|}\big)\|A_\alpha\|_0\|A_\beta\|_0+O(1)\|A_\alpha\|_0|f_{i\overline{\beta}}|_{L^1}.\end{aligned}$$

By the above argument we have $|h^{\alpha\overline{\beta}}O\big(\frac{u_i^3}{|t_i|}\big)\|A_\alpha\|_0\|A_\beta\|_0|=O\big(\frac{u_i^3}{|t_i|}\big)$ and by Lemma 4.3 we have $|h^{\alpha\overline{\beta}}\|A_\alpha\|_0|f_{i\overline{\beta}}|_{L^1}|=O\big(\frac{u_i^3}{|t_i|}\big)$. So the claim is true in this case.

If $\alpha=\beta\leq m$ and $\alpha\neq i$, we have

$$|R_{i\overline{j}\alpha\overline{\alpha}}|\leq\left|\int_X f_{i\overline{j}}e_{\alpha\overline{\alpha}}\ dv\right|+\left|\int_X f_{i\overline{\alpha}}e_{\alpha\overline{j}}\ dv\right|.$$

To estimate the second term in the above formula, we have

$$\left|\int_X f_{i\overline{\alpha}}e_{\alpha\overline{j}}\ dv\right|\leq\|e_{\alpha\overline{j}}\|_0|f_{i\overline{\alpha}}|_{L^1}\leq\|f_{\alpha\overline{j}}\|_0|f_{i\overline{\alpha}}|_{L^1}=O\big(\frac{u_\alpha}{|t_\alpha|}\big)O\big(\frac{u_i^3u_\alpha^3}{|t_it_\alpha|}\big)=O\big(\frac{u_i^3u_\alpha^4}{|t_i||t_\alpha|^2}\big).$$

To estimate the first term, we have

$$\begin{aligned}\left|\int_X f_{i\overline{j}}e_{\alpha\overline{\alpha}}\ dv\right|\leq&\left|\int_X f_{i\overline{j}}\widetilde{e}_{\alpha\overline{\alpha}}\ dv\right|+\left|\int_X f_{i\overline{j}}(e_{\alpha\overline{\alpha}}-\widetilde{e}_{\alpha\overline{\alpha}})\ dv\right|\\ \leq&\left|\int_{\Omega_c^\alpha} f_{i\overline{j}}\widetilde{e}_{\alpha\overline{\alpha}}\ dv\right|+\|e_{\alpha\overline{\alpha}}-\widetilde{e}_{\alpha\overline{\alpha}}\|_0|f_{i\overline{j}}|_{L^1}\\ \leq&\|f_{i\overline{j}}\|_{0,\Omega_c^\alpha}|\widetilde{e}_{\alpha\overline{\alpha}}|_{L^1}+O\big(\frac{u_\alpha^4}{|t_\alpha|^2}\big)O\big(\frac{u_i^3}{|t_i|}\big)=O\big(\frac{u_i^3u_\alpha^3}{|t_i||t_\alpha|^2}\big)\end{aligned}$$

which implies $|h^{\alpha\overline{\alpha}}R_{i\overline{j}\alpha\overline{\alpha}}|=O\big(\frac{u_i^3}{|t_i|}\big)$.

Finally, if $\alpha=\beta=i$, we have

$$|R_{i\overline{j}i\overline{i}}|=2\left|\int_X f_{i\overline{j}}e_{i\overline{i}}\ dv\right|\leq2\|e_{i\overline{i}}\|_0|f_{i\overline{j}}|_{L^1}\leq2\|f_{i\overline{i}}\|_0|f_{i\overline{j}}|_{L^1}=O\big(\frac{u_i^2}{|t_i|^2}\big)O\big(\frac{u_i^3}{|t_i|}\big)=O\big(\frac{u_i^5}{|t_i|^3}\big)$$

which implies $|h^{i\overline{i}}R_{i\overline{j}i\overline{i}}|=O\big(\frac{u_i^2}{|t_i|}\big)$. This proves the third claim.

The second claim can be proved in a similar way. Now we prove the first claim. If $\alpha \neq \beta$ or $\alpha = \beta \geq m+1$ in formula (4.9), we have

$$
\begin{aligned}
|R_{i\bar{i}\alpha\bar{\beta}}| \leq & \left| \int_X f_{i\bar{i}} e_{\alpha\bar{\beta}} \ dv \right| + \left| \int_X f_{i\bar{\beta}} e_{\alpha\bar{i}} \ dv \right| \leq \|e_{\alpha\bar{\beta}}\|_0 |f_{i\bar{i}}|_{L^1} + \left(\int_X |e_{\alpha\bar{i}}|^2 \ dv \int_X |f_{i\bar{\beta}}|^2 \ dv \right)^{\frac{1}{2}} \\
\leq & \|f_{\alpha\bar{\beta}}\|_0 |f_{i\bar{i}}|_{L^1} + \left(\int_X |f_{\alpha\bar{i}}|^2 \ dv \int_X |f_{i\bar{\beta}}|^2 \ dv \right)^{\frac{1}{2}} \leq (\|f_{\alpha\bar{\beta}}\|_0 + \|A_\alpha\|_0 \|A_\beta\|_0) |f_{i\bar{i}}|_{L^1}
\end{aligned}
$$

which implies $|h^{\alpha\bar{\beta}} R_{i\bar{i}\alpha\bar{\beta}}| = O\left(\frac{u_i^3}{|t_i|^2}\right)$.

If $\alpha = \beta \leq m$ and $\alpha \neq i$, we have

$$
|R_{i\bar{i}\alpha\bar{\alpha}}| \leq \left| \int_X e_{i\bar{i}} f_{\alpha\bar{\alpha}} \ dv \right| + \left| \int_X e_{i\bar{\alpha}} f_{\alpha\bar{i}} \ dv \right|.
$$

To estimate the second term in the above inequality, we have

$$
\left| \int_X e_{i\bar{\alpha}} f_{\alpha\bar{i}} \ dv \right| \leq \|e_{i\bar{\alpha}}\|_0 |f_{\alpha\bar{i}}|_{L^1} \leq \|f_{i\bar{\alpha}}\|_0 |f_{\alpha\bar{i}}|_{L^1} = O\left(\frac{u_i u_\alpha}{|t_i t_\alpha|}\right) O\left(\frac{u_i^3 u_\alpha^3}{|t_i t_\alpha|}\right) = O\left(\frac{u_i^4 u_\alpha^4}{|t_i t_\alpha|^2}\right).
$$

To estimate the first term in the above inequality, we have

$$
\begin{aligned}
\left| \int_X e_{i\bar{i}} f_{\alpha\bar{\alpha}} \ dv \right| \leq & \left| \int_X \widetilde{e}_{i\bar{i}} f_{\alpha\bar{\alpha}} \ dv \right| + \left| \int_X (e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) f_{\alpha\bar{\alpha}} \ dv \right| \\
\leq & \left| \int_{\Omega_c^i} \widetilde{e}_{i\bar{i}} f_{\alpha\bar{\alpha}} \ dv \right| + \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 |f_{\alpha\bar{\alpha}}|_{L^1} \\
\leq & \|f_{\alpha\bar{\alpha}}\|_{0,\Omega_c^i} |\widetilde{e}_{i\bar{i}}|_{L^1} + \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 |f_{\alpha\bar{\alpha}}|_{L^1} \\
= & O\left(\frac{u_\alpha^6}{|t_\alpha|^2}\right) O\left(\frac{u_i^3}{|t_i|^2}\right) + O\left(\frac{u_\alpha^3}{|t_\alpha|^2}\right) O\left(\frac{u_i^4}{|t_i|^2}\right) = O\left(\frac{u_i^3 u_\alpha^3}{|t_i t_\alpha|^2}\right).
\end{aligned}
$$

These imply $|h^{\alpha\bar{\alpha}} R_{i\bar{i}\alpha\bar{\alpha}}| = O\left(\frac{u_i^3}{|t_i|^2}\right)$.

Finally, we compute $h^{i\bar{i}} R_{i\bar{i}i\bar{i}}$. Clearly $R_{i\bar{i}i\bar{i}} = 2 \int_X e_{i\bar{i}} f_{i\bar{i}} \ dv$ and

$$
\int_X e_{i\bar{i}} f_{i\bar{i}} \ dv = \int_X \widetilde{e}_{i\bar{i}} \widetilde{f}_{i\bar{i}} \ dv + \int_X \widetilde{e}_{i\bar{i}} (f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}) \ dv + \int_X (e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) f_{i\bar{i}} \ dv.
$$

We also have

$$
\left| \int_X \widetilde{e}_{i\bar{i}} (f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}) \ dv \right| \leq \|f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}\|_0 |\widetilde{e}_{i\bar{i}}|_{L^1} = O\left(\frac{u_i^7}{|t_i|^4}\right)
$$

and

$$
\left| \int_X f_{i\bar{i}} (e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}) \ dv \right| \leq \|e_{i\bar{i}} - \widetilde{e}_{i\bar{i}}\|_0 |f_{i\bar{i}}|_{L^1} = O\left(\frac{u_i^7}{|t_i|^4}\right).
$$

Also, we have $\|\widetilde{e}_{i\bar{i}}\|_{0,\Omega_c^i \setminus \Omega_{c_1}^i} = O\left(\frac{u_i^4}{|t_i|^2}\right)$ and $\|\widetilde{f}_{i\bar{i}}\|_{0,\Omega_c^i \setminus \Omega_{c_1}^i} = O\left(\frac{u_i^4}{|t_i|^2}\right)$. So

$$
\int_X \widetilde{e}_{i\bar{i}} \widetilde{f}_{i\bar{i}} \ dv = \int_{\Omega_{c_1}^i} \widetilde{e}_{i\bar{i}} \widetilde{f}_{i\bar{i}} \ dv + \int_{\Omega_c^i \setminus \Omega_{c_1}^i} \widetilde{e}_{i\bar{i}} \widetilde{f}_{i\bar{i}} \ dv = \frac{3\pi^2}{16} |b_i|^4 u_i (1 + O(u_0)) + O\left(\frac{u_i^8}{|t_i|^4}\right).
$$

By using Corollary 4.1 we have $h^{i\bar{i}} R_{i\bar{i}i\bar{i}} = \frac{3}{4\pi^2} \frac{u_i^2}{|t_i|^2} (1 + O(u_0))$. By combining the above results we have proved this corollary.

□

It is well known that there is a complete asymptotic Poincaré metric ω_p on $\mathcal{M}_g$. We briefly describe it here. Please see [7] for more details.

Let $\overline{M}$ be a compact Kähler manifold of dimension m. Let $Y \subset \overline{M}$ be a divisor of normal crossings and let $M = \overline{M} \setminus Y$. Cover $\overline{M}$ by coordinate charts $U_1, \cdots, U_p, \cdots, U_q$ such that

$(\overline{U}_{p+1} \cup \cdots \cup \overline{U}_q) \cap Y = \Phi$. We also assume that, for each $1 \leq \alpha \leq p$, there is a constant n_α such that $U_\alpha \setminus Y = (\Delta^*)^{n_\alpha} \times \Delta^{m-n_\alpha}$ and on U_α, Y is given by $z_1^\alpha \cdots z_{n_\alpha}^\alpha = 0$. Here Δ is the disk of radius $\frac{1}{2}$ and Δ^* is the punctured disk of radius $\frac{1}{2}$. Let $\{\eta_i\}_{1 \leq i \leq q}$ be the partition of unity subordinate to the cover $\{U_i\}_{1 \leq i \leq q}$. Let ω be a Kähler metric on $\overline{M}$ and let C be a positive constant. Then for C large, the Kähler form

$$\omega_p = C\omega + \sum_{i=1}^{p} \sqrt{-1}\partial\overline{\partial}\Big(\eta_i \log\log \frac{1}{z_1^i \cdots z_{n_i}^i}\Big)$$

defines a complete metric on M with finite volume since on each U_i with $1 \leq i \leq p$, ω_p is bounded from above and below by the local Poincaré metric on U_i. We call this metric the asymptotic Poincaré metric.

As a direct application of the above corollary, we have

Theorem 4.3. *The Ricci metric is equivalent to the asymptotic Poincaré metric. More precisely, there is a positive constant C such that*

$$C^{-1}\omega_p \leq \omega_\tau \leq C\omega_p.$$

Now we estimate the holomorphic sectional curvature of the Ricci metric. We will show that the holomorphic sectional curvature is negative in the degeneration directions and is bounded in other directions. We will need the following estimates on the norms to estimate the error terms.

Lemma 4.6. *Let $f, g \in C^\infty(X, \mathbb{C})$ be smooth functions such that $(\Box + 1)f = g$. Then there is a constant C_0 such that*

(1) $|K_0 f|_{L^2} \leq C_0 |K_0 g|_{L^2}$;
(2) $|K_1 K_0 f|_{L^2} \leq C_0 |K_0 g|_{L^2}$;

Proof. Let $h = |K_0 f|^2$. By using Schwarz inequality, we easily see that the lemma follows from the Bochner formula:

$$\Box h + h + |K_1 K_0 f|^2 = K_0 f \overline{K_0 g} + \overline{K_0 f} K_0 g - |f - g|^2.$$

□

We also need the estimates on the sections $K_0 f_{i\bar{j}}$. We have:

Lemma 4.7. *Let K_0 and K_1 be the Maass operators defined in Section 3. Then*

(1) $\|K_0 f_{i\bar{i}}\|_{0,\Omega_c^i} = O\big(\frac{u_i^2}{|t_i|^2}\big)$ *and* $\|K_0 f_{i\bar{i}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^6}{|t_i|^2}\big)$, *if* $i \leq m$;
(2) $\|K_0 f_{i\bar{j}}\|_0 = O(1)$, *if* $i, j \geq m+1$;
(3) $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_i u_j^3}{|t_i t_j|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^j} = O\big(\frac{u_i^3 u_j}{|t_i t_j|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,X\setminus(\Omega_c^i \cup \Omega_c^j)} = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$, *if* $i, j \leq m$ *and* $i \neq j$;
(4) $\|K_0 f_{i\bar{j}}\|_{0,\Omega_c^i} = O\big(\frac{u_i}{|t_i|}\big)$ *and* $\|K_0 f_{i\bar{j}}\|_{0,X\setminus\Omega_c^i} = O\big(\frac{u_i^3}{|t_i|}\big)$, *if* $i \leq m$ *and* $j \geq m+1$;
(5) $\|f_{i\bar{i}} - \widetilde{f}_{i\bar{i}}\|_1 = O\big(\frac{u_i^4}{|t_i|^2}\big)$, *if* $i \leq m$.

This lemma can be proved by using similar methods as we used in the proof of Lemma 4.3 together with direct computations. So are the following L^1 and L^2 estimates:

Lemma 4.8. *Let $P = K_1 K_0$ be the operator defined Section 3. We have*

(1) $|f_{i\bar{i}}|_{L^2}^2 = O\big(\frac{u_i^5}{|t_i|^4}\big)$, *if* $i \leq m$;
(2) $|K_0 f_{i\bar{i}}|_{L^2}^2 = O\big(\frac{u_i^5}{|t_i|^4}\big)$, *if* $i \leq m$;
(3) $|K_0 f_{i\bar{j}}|_{L^2}^2 = O\big(\frac{u_i^3 u_j^3}{|t_i t_j|^2}\big)$, *if* $i, j \leq m$ *and* $i \neq j$;
(4) $|K_0 f_{i\bar{j}}|_{L^2}^2 = O\big(\frac{u_i^3}{|t_i|^2}\big)$, *if* $i \leq m$ *and* $j \geq m+1$;

(5) $|K_0 f_{i\bar{j}}|^2_{L^2} = O(1)$, *if* $i, j \geq m+1$;

(6) $|P(\widetilde{e}_{i\bar{i}})|_{L^1} = O\big(\frac{u_i^3}{|t_i|^2}\big)$, *if* $i \leq m$.

To estimate the curvature of the Ricci metric by using formula (3.30), we first expand the term $\int_X Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}}\, dv$. A simple computation shows that

Lemma 4.9. *We have*

$$\begin{aligned}\int_X Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}}\, dv = & -\int_X f_{k\bar{l}}(K_0 e_{i\bar{j}} \overline{K}_0 e_{\alpha\bar{\beta}} + \overline{K}_0 e_{i\bar{j}} K_0 e_{\alpha\bar{\beta}})\, dv \\ & - \int_X (\Box e_{i\bar{j}} K_0 e_{\alpha\bar{\beta}} \overline{K}_0 e_{k\bar{l}} + \Box e_{\alpha\bar{\beta}} K_0 e_{i\bar{j}} \overline{K}_0 e_{k\bar{l}})\, dv.\end{aligned}$$

To estimate the holomorphic sectional curvature, in formula (3.30) we let $i = j = k = l$. We decompose $\widetilde{R}_{i\bar{i}i\bar{i}}$ into two parts:

$$\widetilde{R}_{i\bar{i}i\bar{i}} = G_1 + G_2$$

where G_1 consists of those terms in the right hand side of (3.30) with all indices α, β, γ, δ, p and q equal to i and $G_2 = \widetilde{R}_{i\bar{i}i\bar{i}} - G_1$ consists of those terms in (3.30) where, in each term, at least one of the indices α, β, γ, δ, p or q is not i. If $i \leq m$, the leading term is G_1 which is given by

$$\begin{aligned}G_1 = & 24 h^{i\bar{i}} \int_X (\Box + 1)^{-1}(\xi_i(e_{i\bar{i}})) \overline{\xi}_i(e_{i\bar{i}})\, dv \\ & + 6 h^{i\bar{i}} \int_X Q_{i\bar{i}}(e_{i\bar{i}}) e_{i\bar{i}}\, dv \\ & - 36 \tau^{i\bar{i}} (h^{i\bar{i}})^2 \left| \int_X \xi_i(e_{i\bar{i}}) e_{i\bar{i}}\, dv \right|^2 \\ & + \tau_{i\bar{i}} h^{i\bar{i}} R_{i\bar{i}i\bar{i}}.\end{aligned} \tag{4.10}$$

The main theorem of this section is the following estimate of the holomorphic sectional curvature of the Ricci metric.

Theorem 4.4. *Let* $X_0 \in \overline{\mathcal{M}_g} \backslash \mathcal{M}_g$ *be a codimension* m *point and let* $(t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ *be the pinching coordinates at* X_0 *where* $t_1, \cdots, t_m$ *correspond to the degeneration directions. Then the holomorphic sectional curvature is negative in the degeneration directions and is bounded in the non-degeneration directions. More precisely, there is a* $\delta > 0$ *such that, if* $|(t,s)| < \delta$, *then*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4 |t_i|^4}(1 + O(u_0)) > 0 \tag{4.11}$$

if $i \leq m$ *and*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \tag{4.12}$$

if $i \geq m+1$.

Furthermore, on $\mathcal{M}_g$, *the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.*

Proof. We first compute the asymptotics of the holomorphic sectional curvature. By Lemma 4.9 we know that

$$\int_X Q_{i\bar{i}}(e_{i\bar{i}}) e_{i\bar{i}}\, dv = \int_X |K_0 e_{i\bar{i}}|^2 (2 e_{i\bar{i}} - 4 f_{i\bar{i}})\, dv.$$

By (4.10) we have

$$
\begin{aligned}
G_1 =&24h^{i\bar{i}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\ dv + 6h^{i\bar{i}}\int_X |K_0 e_{i\bar{i}}|^2(2e_{i\bar{i}}-4f_{i\bar{i}})\ dv \\
&-36\tau^{i\bar{i}}(h^{i\bar{i}})^2\left|\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv\right|^2 + \tau_{i\bar{i}}h^{i\bar{i}}R_{i\bar{i}i\bar{i}}.
\end{aligned}
\tag{4.13}
$$

We first consider the degeneration directions. Assume $i \le m$. In this case G_1 is the leading term. We have the following lemma.

Lemma 4.10. *If $i \le m$, then $|G_2| = O\big(\frac{u_i^5}{|t_i|^4}\big)$.*

Proof. The lemma follows from a case by case check. We will prove it in the appendix. $\square$

Now we go back to the proof of Theorem 4.4. We compute each term of G_1. By the proof of Corollary 4.2 we know that $h^{i\bar{i}}R_{i\bar{i}i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$. So we have

$$
\tau_{i\bar{i}}h^{i\bar{i}}R_{i\bar{i}i\bar{i}} = \left(\frac{3u_i^2}{4\pi^2|t_i|^2}\right)^2(1+O(u_0)) = \frac{9u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)).
\tag{4.14}
$$

Now we compute the second term. We have

$$
\begin{aligned}
&\int_X |K_0 e_{i\bar{i}}|^2(2e_{i\bar{i}}-4f_{i\bar{i}})\ dv \\
=&\int_X |K_0 \widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv + \int_X (|K_0 e_{i\bar{i}}|^2 - |K_0 \widetilde{e}_{i\bar{i}}|^2)(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv \\
&+\int_X |K_0 e_{i\bar{i}}|^2(2(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})-4(f_{i\bar{i}}-\widetilde{f}_{i\bar{i}}))\ dv.
\end{aligned}
\tag{4.15}
$$

For the second term in the above equation, we have

$$
\begin{aligned}
&\left|\int_X (|K_0 e_{i\bar{i}}|^2 - |K_0 \widetilde{e}_{i\bar{i}}|^2)(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv\right| \le \||K_0 e_{i\bar{i}}|^2 - |K_0 \widetilde{e}_{i\bar{i}}|^2\|_0 \int_X (2|\widetilde{e}_{i\bar{i}}|+4|\widetilde{f}_{i\bar{i}}|)\ dv \\
\le&\||K_0 e_{i\bar{i}}| + |K_0 \widetilde{e}_{i\bar{i}}|\|_0 \|K_0(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})\|_0 \int_X (2|\widetilde{e}_{i\bar{i}}|+4|\widetilde{f}_{i\bar{i}}|)\ dv = O\big(\frac{u_i^2}{|t_i|^2}\big)O\big(\frac{u_i^4}{|t_i|^2}\big)O\big(\frac{u_i^3}{|t_i|^2}\big) = O\big(\frac{u_i^9}{|t_i|^6}\big).
\end{aligned}
$$

For the second term in the above equation, we have

$$
\begin{aligned}
&\left|\int_X |K_0 e_{i\bar{i}}|^2(2(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})-4(f_{i\bar{i}}-\widetilde{f}_{i\bar{i}}))\ dv\right| \le C_0\|K_0 e_{i\bar{i}}\|_0^2(2\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_0 + 4\|f_{i\bar{i}}-\widetilde{f}_{i\bar{i}}\|_0) \\
=&O\big(\frac{u_i^4}{|t_i|^4}\big)O\big(\frac{u_i^4}{|t_i|^2}\big) = O\big(\frac{u_i^8}{|t_i|^6}\big).
\end{aligned}
$$

So we get

$$
\begin{aligned}
&\int_X |K_0 e_{i\bar{i}}|^2(2e_{i\bar{i}}-4f_{i\bar{i}})\ dv = \int_X |K_0 \widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv + O\big(\frac{u_i^8}{|t_i|^6}\big) \\
=&\int_{\Omega_{c_1}^i} |K_0 \widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv + \int_{\Omega_c^i\setminus\Omega_{c_1}^i} |K_0 \widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv + O\big(\frac{u_i^8}{|t_i|^6}\big).
\end{aligned}
\tag{4.16}
$$

We also have the estimate

$$
\left|\int_{\Omega_c^i\setminus\Omega_{c_1}^i} |K_0 \widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}}-4\widetilde{f}_{i\bar{i}})\ dv\right| \le C_0\|K_0\widetilde{e}_{i\bar{i}}\|_0^2(\|\widetilde{e}_{i\bar{i}}\|_{0,\Omega_c^i\setminus\Omega_{c_1}^i} + \|\widetilde{f}_{i\bar{i}}\|_{0,\Omega_c^i\setminus\Omega_{c_1}^i}) = O\big(\frac{u_i^8}{|t_i|^6}\big).
$$

A direct computation shows that

$$\int_{\Omega_{c_1}^i} |K_0\widetilde{e}_{i\bar{i}}|^2(2\widetilde{e}_{i\bar{i}} - 4\widetilde{f}_{i\bar{i}})\ dv = -\frac{3u_i^7}{64\pi^4|t_i|^6}(1+O(u_0)).$$

So

$$6h^{i\bar{i}}\int_X |K_0 e_{i\bar{i}}|^2(2e_{i\bar{i}} - 4f_{i\bar{i}})\ dv = -\frac{9u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)). \tag{4.17}$$

Now we compute the third term. We have

$$\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv = \int_X \xi_i(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv + \int_X \xi_i(\widetilde{e}_{i\bar{i}})(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})\ dv + \int_X \xi_i(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})e_{i\bar{i}}\ dv. \tag{4.18}$$

By using the same method as above, we obtain

$$\begin{aligned}\left|\int_X \xi_i(\widetilde{e}_{i\bar{i}})(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})\ dv\right| &\le C_0\|\xi_i(\widetilde{e}_{i\bar{i}})\|_0\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_0 \le C_0\|A_i\|_0\|K_1K_0(\widetilde{e}_{i\bar{i}})\|_0\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_0\\ &\le C_0\|A_i\|_0\|\widetilde{e}_{i\bar{i}}\|_2\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_0 = O\big(\frac{u_i}{|t_i|}\big)O\big(\frac{u_i^2}{|t_i|^2}\big)O\big(\frac{u_i^4}{|t_i|^2}\big) = O\big(\frac{u_i^7}{|t_i|^5}\big)\end{aligned}$$

and

$$\begin{aligned}\left|\int_X \xi_i(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})e_{i\bar{i}}\ dv\right| &\le \|\xi_i(e_{i\bar{i}}-\widetilde{e}_{i\bar{i}})\|_0\int_X e_{i\bar{i}}\ dv \le \|A_i\|_0\|e_{i\bar{i}}-\widetilde{e}_{i\bar{i}}\|_2 h_{i\bar{i}}\\ &\le \|A_i\|_0\|f_{i\bar{i}}-\widetilde{f}_{i\bar{i}}\|_1 h_{i\bar{i}} = O\big(\frac{u_i}{|t_i|}\big)O\big(\frac{u_i^4}{|t_i|^2}\big)O\big(\frac{u_i^3}{|t_i|^2}\big) = O\big(\frac{u_i^8}{|t_i|^5}\big)\end{aligned}$$

and

$$\left|\int_{\Omega_c^i\setminus\Omega_{c_1}^i} \xi_i(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv\right| \le C_0\|\xi_i(\widetilde{e}_{i\bar{i}})\|_0\|\widetilde{e}_{i\bar{i}}\|_{0,\Omega_c^i\setminus\Omega_{c_1}^i} = O\big(\frac{u_i^7}{|t_i|^5}\big).$$

By putting the above results together, we get

$$\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv = \int_{\Omega_{c_1}^i} \xi_i(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv + O\big(\frac{u_i^7}{|t_i|^5}\big).$$

On $\Omega_{c_1}^i$ we have

$$\xi_i(\widetilde{e}_{i\bar{i}}) = -\frac{z_i}{\bar{z}_i}\sin^2\tau_i\overline{b_i}P(\widetilde{e}_{i\bar{i}}) - \frac{z_i}{\bar{z}_i}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}}).$$

However, we have $\|\frac{z_i}{\bar{z}_i}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}})\|_{0,\Omega_{c_1}^i} = O\big(\frac{u_i^5}{|t_i|^3}\big)$ which implies

$$\left|\int_{\Omega_{c_1}^i} \frac{z_i}{\bar{z}_i}\sin^2\tau_i\overline{p_i}P(\widetilde{e}_{i\bar{i}})\widetilde{e}_{i\bar{i}}\ dv\right| = O\big(\frac{u_i^8}{|t_i|^5}\big).$$

A direct computation shows that

$$\int_{\Omega_{c_1}^i} -\frac{z_i}{\bar{z}_i}\sin^2\tau_i\overline{b_i}P(\widetilde{e}_{i\bar{i}}))\widetilde{e}_{i\bar{i}}\ dv = -\frac{u_i^6}{32\pi^3|t_i|^4 t_i}(1+O(u_0))$$

which implies

$$\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv = -\frac{u_i^6}{32\pi^3|t_i|^4 t_i}(1+O(u_0)).$$

So we obtain

$$36\tau^{i\bar{i}}(h^{i\bar{i}})^2\left|\int_X \xi_i(e_{i\bar{i}})e_{i\bar{i}}\ dv\right|^2 = \frac{3u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)). \tag{4.19}$$

Now we estimate the first term. We have

$$\int_X T\xi_i(e_{i\bar i})\overline{\xi_i}(e_{i\bar i})\,dv = \int_X T\xi_i(\widetilde{e}_{i\bar i})\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv + \int_X T\xi_i(e_{i\bar i}-\widetilde{e}_{i\bar i})\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv$$
$$+\int_X T\xi_i(e_{i\bar i})\overline{\xi_i}(e_{i\bar i}-\widetilde{e}_{i\bar i})\,dv.$$

By using the same method we can get

$$\begin{aligned}\left|\int_X T\xi_i(e_{i\bar i}-\widetilde{e}_{i\bar i})\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv\right| \le& C_0\|T\xi_i(e_{i\bar i}-\widetilde{e}_{i\bar i})\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar i})\|_0 \le C_0\|\xi_i(e_{i\bar i}-\widetilde{e}_{i\bar i})\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar i})\|_0\\ =&O\Big(\frac{u_i^5}{|t_i|^3}\Big)O\Big(\frac{u_i^3}{|t_i|^3}\Big)=O\Big(\frac{u_i^8}{|t_i|^6}\Big).\end{aligned}$$

Similarly,

$$\left|\int_X T\xi_i(e_{i\bar i})\overline{\xi_i}(e_{i\bar i}-\widetilde{e}_{i\bar i})\,dv\right| = O\Big(\frac{u_i^8}{|t_i|^6}\Big).$$

So we have

$$\int_X T\xi_i(e_{i\bar i})\overline{\xi_i}(e_{i\bar i})\,dv = \int_X T\xi_i(\widetilde{e}_{i\bar i})\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv + O\Big(\frac{u_i^8}{|t_i|^6}\Big).$$

To estimate $T\xi_i(\widetilde{e}_{i\bar i})$, we introduce another approximation function. Pick $c_2 < c_1$ and let $\eta_1 \in C^\infty(\mathbb{R},[0,1])$ be the cut-off function defined by

$$\eta_1 = \begin{cases}\eta_1(x)=1, & x\le \log c_2;\\ \eta_1(x)=0, & x\ge \log c_1;\\ 0<\eta_1(x)<1, & \log c_2 < x < \log c_1.\end{cases} \tag{4.20}$$

For $i \le m$ define the function d_i by

$$d_i(z)=\begin{cases}-\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i}, & z\in\Omega^i_{c_2};\\ (-\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i})\eta_1(\log r_i), & z\in\Omega^i_{c_1}\text{ and } c_2<r_i<c_1;\\ (-\frac{1}{8}\sin^2\tau_i\cos 2\tau_i|b_i|^2\overline{b_i})\eta_1(\log\rho_i-\log r_i), & z\in\Omega^i_{c_1}\text{ and } c_1^{-1}\rho_i<r_i<c_2^{-1}\rho_i;\\ 0, & z\in X\setminus\Omega^i_{c_1}.\end{cases}$$

A simple computation shows that

$$\|\xi_i(\widetilde{e}_{i\bar i})-(\Box+1)d_i\|_0 = O\Big(\frac{u_i^5}{|t_i|^3}\Big)$$

which implies

$$\|T\xi_i(\widetilde{e}_{i\bar i})-d_i\|_0 = O\Big(\frac{u_i^5}{|t_i|^3}\Big).$$

So

$$\int_X T\xi_i(\widetilde{e}_{i\bar i})\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv = \int_X d_i\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv + \int_X (T\xi_i(\widetilde{e}_{i\bar i})-d_i)\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv.$$

We have the estimate

$$\left|\int_X (T\xi_i(\widetilde{e}_{i\bar i})-d_i)\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv\right| \le C_0\|T\xi_i(\widetilde{e}_{i\bar i})-d_i\|_0\|\overline{\xi_i}(\widetilde{e}_{i\bar i})\|_0 = O\Big(\frac{u_i^8}{|t_i|^6}\Big)$$

which implies

$$\int_X T\xi_i(e_{i\bar i})\overline{\xi_i}(e_{i\bar i})\,dv = \int_X d_i\overline{\xi_i}(\widetilde{e}_{i\bar i})\,dv + O\Big(\frac{u_i^8}{|t_i|^6}\Big).$$

We also have

$$d_i\overline{\xi_i}(\widetilde{e}_{i\bar i}) = -d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i b_i\overline{P}(\widetilde{e}_{i\bar i}) - d_i\frac{\overline{z_i}}{z_i}\sin^2\tau_i p_i\overline{P}(\widetilde{e}_{i\bar i}).$$

Since $\|d_i \frac{\overline{z_i}}{z_i} \sin^2 \tau_i p_i \overline{P}(\widetilde{e}_{i\bar{i}})\|_0 = O\big(\frac{u_i^8}{|t_i|^6}\big)$ and $\|d_i \frac{\overline{z_i}}{z_i} \sin^2 \tau_i b_i \overline{P}(\widetilde{e}_{i\bar{i}})\|_{0,\Omega^i_{c_1}\setminus \Omega^i_{c_2}} = O\big(\frac{u_i^8}{|t_i|^6}\big)$, we get

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\, dv = \int_{\Omega^i_{c_2}} d_i \overline{\xi_i}(\widetilde{e}_{i\bar{i}})\, dv + O\big(\frac{u_i^8}{|t_i|^6}\big).$$

A direct computation shows that

$$\int_X T\xi_i(e_{i\bar{i}})\overline{\xi_i}(e_{i\bar{i}})\, dv = \frac{3u_i^7}{256\pi^4|t_i|^6}(1+O(u_0))$$

which implies

$$24h^{i\bar{i}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\, dv = \frac{9u_i^4}{16\pi^4|t_i|^4}(1+O(u_0)). \tag{4.21}$$

By combining formulas (5.3), (4.17), (4.19) and (4.14) we obtain

$$G_1 = \frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0)).$$

Together with Lemma 4.10 we proved formula (4.11). The formula (4.12) can be proved using similar method with a case by case like the proof of Lemma 4.10.

Now we give a weak estimate on the full curvature of the Ricci metric. Let

(1) $\Lambda_i = \frac{u_i}{|t_i|}$ if $i \le m$;
(2) $\Lambda_i = 1$ if $i \ge m+1$.

We can check the following estimates by using the methods in the proof of Lemma 4.10. We have

$$\widetilde{R}_{i\bar{j}k\bar{l}} = O(1) \tag{4.22}$$

if $i,j,k,l \ge m+1$ and

$$\widetilde{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i\Lambda_j\Lambda_k\Lambda_l)O(u_0) \tag{4.23}$$

if at least one of these indices i,j,k,l is less than or equal to m and they are not all equal to each other.

Now we prove the boundedness of the curvatures. For the holomorphic sectional curvature, from (4.11) and (4.12) and Corollary 4.2, it is clear that there is a constant $C_0 > 1$ depending on X_0 and δ such that if $|(t,s)| \le \delta$, then

(1) $C_0^{-1}\tau_{i\bar{i}}^2 \le \widetilde{R}_{i\bar{i}i\bar{i}} \le C_0\tau_{i\bar{i}}^2$, if $i \le m$;
(2) $|\widetilde{R}_{i\bar{i}i\bar{i}}| \le C_0\tau_{i\bar{i}}^2$, if $i \ge m+1$.

We cover the divisor $Y = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ by such open coordinate charts. Since Y is compact, we can pick finitely many such coordinate charts $\Xi_1, \cdots, \Xi_q$ such that $Y \subset \bigcup_{s=1}^q \Xi_s$. Clearly there is an open neighborhood N of Y such that $\overline{N} \subset \bigcup_{s=1}^q \Xi_s$. From formulas (4.22), (4.23) and the above argument, we know that the holomorphic sectional curvature of τ is bounded from above and below on N. However, $\mathcal{M}_g \setminus N$ is a compact set of $\mathcal{M}_g$, so the holomorphic sectional curvature is also bounded on $\mathcal{M}_g \setminus N$ which implies the holomorphic sectional curvature is bounded on $\mathcal{M}_g$.

The bisectional curvature and the Ricci curvature of the Ricci metric can be proved to be bounded by using (4.22), (4.23) and a similar argument as above, together with the covering and compactness argument. This finishes the proof.

□

Remark 4.4. The estimates of the bisectional curvature and the Ricci curvature are not optimal. A sharper estimate will be given in our next paper [6].

5. The perturbed Ricci metric and its curvatures

In this section we introduce another new metric, the perturbed Ricci metric. This metric is obtained by adding a constant multiple of the Weil-Petersson metric to the Ricci metric. By doing this we construct a natural complete metric whose holomorphic sectional curvature is negatively bounded. We will see that the holomorphic sectional curvature of the perturbed Ricci metric near an interior point of the moduli space is dominated by the curvature of the large constant multiple of the Weil-Petersson metric. Similar argument holds for the holomorphic sectional curvature of the perturbed Ricci metric in the non-degenerate directions near a boundary point.

Definition 5.1. *For any constant $C > 0$, we call the metric*

$$\widetilde{\tau}_{i\overline{j}} = \tau_{i\overline{j}} + Ch_{i\overline{j}}$$

the perturbed Ricci metric with constant C.

We first give the curvature formula of the perturbed Ricci metric. We use $P_{i\overline{j}k\overline{l}}$ to denote the curvature tensor of the perturbed Ricci metric.

Theorem 5.1. *Let $s_1, \cdots, s_n$ be local holomorphic coordinates at $s \in M_g$. Then at s, we have*

$$\begin{aligned}P_{i\overline{j}k\overline{l}} =& h^{\alpha\overline{\beta}}\left\{\sigma_1\sigma_2\int_{X_s}\left\{(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+(\Box+1)^{-1}(\xi_k(e_{i\overline{j}}))\overline{\xi}_\beta(e_{\alpha\overline{l}})\right\}\ dv\right\}\\&+h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}Q_{k\overline{l}}(e_{i\overline{j}})e_{\alpha\overline{\beta}}\ dv\right\}\\&-\widetilde{\tau}^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\left\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right\}\left\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\overline{j}})e_{\gamma\overline{\delta}})\ dv\right\}\\&+\tau_{p\overline{j}}h^{p\overline{q}}R_{i\overline{q}k\overline{l}}+CR_{i\overline{j}k\overline{l}}.\end{aligned}\tag{5.1}$$

Proof. Let $s_1, \cdots, s_n$ be normal coordinates at a point $s \in \mathcal{M}_g$ with respect to the Weil-Petersson metric. By formula (3.16), at the point s we have

$$\begin{aligned}\partial_k\widetilde{\tau}_{i\overline{j}} &= \partial_k\tau_{i\overline{j}} + C\partial_k h_{i\overline{j}} = h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}(\xi_k(e_{i\overline{j}})e_{\alpha\overline{\beta}})\ dv\right\}+\tau_{p\overline{j}}\Gamma_{ik}^p + C\partial_k h_{i\overline{j}}\\&= h^{\alpha\overline{\beta}}\left\{\sigma_1\int_{X_s}(\xi_k(e_{i\overline{j}})e_{\alpha\overline{\beta}})\ dv\right\}\end{aligned}\tag{5.2}$$

since $\Gamma_{ik}^p = \partial_k h_{i\overline{j}} = 0$ at this point. Now at s the curvature of the Weil-Petersson metric is

$$R_{i\overline{j}k\overline{l}} = \partial_{\overline{l}}\partial_k h_{i\overline{j}}.$$

The theorem follows from formulas (3.5), (5.2) and (3.36).

□

Now we estimate the curvature of the perturbed Ricci metric using formula (5.1). The following two linear algebra lemmas will be used to handle the inverse matrix $\widetilde{\tau}^{i\overline{j}}$ near an interior point and a boundary point.

Lemma 5.1. *Let D be a neighborhood of 0 in $\mathbb{C}^n$ and let A and B be two positive definite $n \times n$ Hermitian matrix functions on D such that they are bounded from above and below on D and each entry of them are bounded. Then each entry of the inverse matrix $(A + CB)^{-1} = O(C^{-1})$ when C is very large.*

Proof. Consider the determinant $\det(A + CB)$. It is a polynomial of C of degree n and the coefficient of the leading term is $\det(B)$ which is bounded from below. All other coefficients are bounded since they only depend on the entries of A and B. So we can pick C large such that

$\det(A+CB) \geq \frac{1}{2}\det(B)C^n$. Now the determinant of the (i,j)-minor of $A+CB$ is a polynomial of C of degree at most $n-1$ and the coefficients are bounded since they only depend on the entries of A and B. From the fact that the (i,j)-entry is the quotient of the determinant of the (i,j)-minor and the determinant of the matrix $A+CB$, the lemma follows directly.

□

Lemma 5.2. *Let $X_0 \in \overline{\mathcal{M}}_g$ be a codimension m boundary point and let $(t_1, \cdots, s_n)$ be the pinching coordinates near X_0. Then for $|(t,s)| < \delta$ with δ small, we have that, for any $C > 0$,*

(1) $0 < \widetilde{\tau}^{i\bar{i}} < \tau^{i\bar{i}}$ *for all i;*
(2) $\widetilde{\tau}^{i\bar{j}} = O(|t_i t_j|)$, *if $i, j \leq m$ and $i \neq j$;*
(3) $\widetilde{\tau}^{i\bar{j}} = O(|t_i|)$, *if $i \leq m$ and $j \geq m+1$;*
(4) $\widetilde{\tau}^{i\bar{j}} = O(1)$, *if $i, j \geq m+1$.*

Furthermore, the bounds in the last three claims are independent of the choice of C.

Proof. The first claim is a general fact of linear algebra. To prove the last three claims, we denote the submatrices $(\tau_{i\bar{j}})_{i,j\geq m+1}$ and $(h_{i\bar{j}})_{i,j\geq m+1}$ by A and B. These two matrices represent the non-degenerate directions of the Ricci metric and the Weil-Petersson metric respectively. By the work of Masur, we know that the matrix B can by extended to the boundary non-degenerately. This implies that B has a positive lower bound. By Corollary (4.1) we know that B is bounded from above. Now by the work of Wolpert, since $\omega_\tau \geq \widetilde{C}\omega_{WP}$ where $\widetilde{C}$ only depend on the genus of the Riemann surface, we know that A has a positive lower bound. By Corollary 4.2 we know that A is bounded from above. So both matrices A and B are bounded from above and below and all their entries are bounded as long as $|(t,s)| \leq \delta$.

By Corollary 4.1 and Corollary 4.2 we know that

$$(\widetilde{\tau}_{i\bar{j}}) = \begin{pmatrix} \Upsilon & \Psi \\ \overline{\Psi}^T & A+CB \end{pmatrix}$$

where Υ is an $m \times m$ matrix given by

$$\Upsilon = \begin{pmatrix} \frac{u_1^2}{|t_1|^2}(\frac{3}{4\pi^2} + \frac{Cu_1}{2})(1+O(u_0)) & \cdots & \frac{u_1^2 u_m^2}{|t_1 t_m|}(O(u_0) + CO(u_1 u_m)) \\ \vdots & \vdots & \vdots \\ \frac{u_1^2 u_m^2}{|t_1 t_m|}(O(u_0) + CO(u_1 u_m)) & \cdots & \frac{u_m^2}{|t_m|^2}(\frac{3}{4\pi^2} + \frac{Cu_m}{2})(1+O(u_0)) \end{pmatrix}$$

which represent the degenerate directions of the perturbed Ricci metric and Ψ is an $m \times (n-m)$ matrix given by

$$\Psi = \begin{pmatrix} \frac{u_1^2}{|t_1|}(O(1) + CO(u_1)) & \cdots & \frac{u_1^2}{|t_1|}(O(1) + CO(u_1)) \\ \vdots & \vdots & \vdots \\ \frac{u_m^2}{|t_m|}(O(1) + CO(u_m)) & \cdots & \frac{u_m^2}{|t_m|}(O(1) + CO(u_m)) \end{pmatrix}$$

which represents the mixed directions of the perturbed Ricci metric.

A direct computation shows that

$$\det \widetilde{\tau} = \{\prod_{i=1}^m \frac{u_i^2}{|t_i|^2}(\frac{3}{4\pi^2} + \frac{Cu_i}{2})\} \det(A+CB)(1+O(u_0))$$

where the $O(u_0)$ term is independent of C. Let Φ_{ij} be the (i,j)-minor of $(\widetilde{\tau}_{i\bar{j}})$ obtained by deleting the i-th row and j-th column of $(\widetilde{\tau}_{i\bar{j}})$. By using the fact that

$$|\widetilde{\tau}^{i\bar{j}}| = \left|\frac{\det \Phi_{ij}}{\det \widetilde{\tau}}\right|$$

the lemma follows from a direct computation of the determinant of Φ_{ij}.

$\square$

Now we prove the main theorem of this section.

Theorem 5.2. *For a suitable choice of positive constant C, the perturbed Ricci metric $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ is complete and its holomorphic sectional curvatures are negative and bounded from above and below by negative constants. Furthermore, the Ricci curvature of the perturbed Ricci metric is bounded from above and below.*

Proof. It is clear that the metric $\widetilde{\tau}_{i\bar{j}}$ is complete as long as $C \geq 0$ since it is greater than the Ricci metric which is complete.

Now we estimate the holomorphic sectional curvature. We first show that, for any codimension m point $X_0 \in \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$, there are constants $C_0, \delta > 0$ such that, if $(t,s) = (t_1, \cdots, t_m, s_{m+1}, \cdots, s_n)$ is the pinching coordinates at p with $|(t,s)| < \delta$ and $C \geq C_0$, the holomorphic sectional curvature of the metric $\widetilde{\tau}$ is negative. We first consider the degeneration directions. Let $i = j = k = l \leq m$. As in the proof of Theorem 4.4, we let

$$
\begin{aligned}
\widetilde{G}_1 = & 24h^{i\bar{i}} \int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{i\bar{i}})\ dv + 6h^{i\bar{i}} \int_X |K_0 e_{i\bar{i}}|^2 (2e_{i\bar{i}} - 4f_{i\bar{i}})\ dv \\
& - 36\widetilde{\tau}^{i\bar{i}}(h^{i\bar{i}})^2 \left| \int_X \xi_i(e_{i\bar{i}}) e_{i\bar{i}}\ dv \right|^2 + \tau_{i\bar{i}} h^{i\bar{i}} R_{i\bar{i}i\bar{i}}
\end{aligned}
\tag{5.3}
$$

and $\widetilde{G}_2$ be the summation of those terms in (5.1) in which at least one of the indices $p, q, \alpha, \beta, \gamma, \delta$ is not i. We have $P_{i\bar{i}i\bar{i}} = \widetilde{G}_1 + \widetilde{G}_2 + CR_{i\bar{i}i\bar{i}}$. We notice here that we can use Lemma 5.2 instead of Corollary 4.2 in the proof of Lemma 4.10 without changing any estimate. This implies that $|\widetilde{G}_2| = O\big(\frac{u_i^5}{|t_i|^4}\big)$. By the proof of Theorem 4.4 we have

$$
\widetilde{G}_1 = \left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4} \big(1 + \frac{2\pi^2 C u_i}{3}\big)^{-1} \right) \frac{u_i^4}{|t_i|^4} (1 + O(u_0))
\tag{5.4}
$$

which implies

$$
P_{i\bar{i}i\bar{i}} = \left(\left(\frac{9}{16\pi^4} - \frac{3}{16\pi^4} \big(1 + \frac{2\pi^2 C u_i}{3}\big)^{-1} \right) \frac{u_i^4}{|t_i|^4} + \frac{3C}{8\pi^2} \frac{u_i^5}{|t_i|^4} \right) (1 + O(u_0)) > 0
\tag{5.5}
$$

as long as δ is small enough. Furthermore, $P_{i\bar{i}i\bar{i}}$ is bounded above and below by constant multiple of $\widetilde{\tau}_{i\bar{i}}^2$ where the constants may depend on C. However, when C is fixed, the constants are universal if δ is small enough.

Now we let $i = j = k = l \geq m+1$. By the proof of Theorem 4.4 and Lemma 5.2 we know that $P_{i\bar{i}i\bar{i}} = O(1) + CR_{i\bar{i}i\bar{i}}$. We also know that $R_{i\bar{i}i\bar{i}} > 0$ has a positive lower bound. Again, by using the extension theorem of Masur, we can choose C_0 large enough such that, when $C \geq C_0$, we have $P_{i\bar{i}i\bar{i}} > 0$. Furthermore, $P_{i\bar{i}i\bar{i}}$ is bounded from above and below by constant multiple of $\widetilde{\tau}_{i\bar{i}}^2$ where the constants may depend on C, m, n, X_0 and the choice of $\nu_{m+1}, \cdots, \nu_n$ if δ is small enough. We also have estimates similar to (4.22) and (4.23):

$$
P_{i\bar{j}k\bar{l}} = O(1) + CR_{i\bar{j}k\bar{l}}
\tag{5.6}
$$

if $i, j, k, l \geq m+1$ and

$$
P_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0) + CR_{i\bar{j}k\bar{l}}
\tag{5.7}
$$

if at least one of these indices i, j, k, l is less than or equal to m and they are not all equal to each other. So we can choose δ small such that, if $|(t,s)| \leq \delta$, then the holomorphic sectional curvature is bounded from above and below by negative constants which may depend on C.

Now we consider the interior points. Fix a point $p \in \mathcal{M}_g$ and a small neighborhood D of p such that $\overline{D} \subset \mathcal{M}_g$. Since the Ricci metric and Weil-Petersson metric are uniformly bounded in

$\overline{D}$, we have $P_{\bar{i}\bar{i}i\bar{i}} = O(1) + CR_{\bar{i}\bar{i}i\bar{i}}$. Using a similar argument as above, we can choose a C_0 such that, when $C > C_0$, the holomorphic sectional curvature is bounded from above and below by negative constants which may depend on C.

Since the divisor $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is compact, we can find finitely many boundary charts of $\mathcal{M}_g$ described above such that the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by two negative constants which depend on C on these charts. Furthermore, there is a neighborhood N of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ in $\overline{\mathcal{M}}_g$ such that $\overline{N}$ is contained in the union of these charts. It is clear that we can find a constant C_1 such that on N, the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by negative constants when $C \geq C_1$.

Also, since the set $\mathcal{M}_g \setminus N$ is compact, by the above argument, we can find finitely many interior charts described above such that their union covers $\mathcal{M}_g \setminus N$ and a constant C_2, such that the holomorphic sectional curvature of $\widetilde{\tau}$ is pinched by negative constants when $C > C_2$. Again, the bounds may depend on C.By taking a constant $C > \max\{C_1, C_2\}$, we have proved the first part of the theorem. The Ricci curvature can be estimated in a similar way as we did in the proof of Theorem 4.4 together with Lemma 5.1 and 5.2.

□

Remark 5.1. By using the negativity of the Ricci curvature of the Weil-Petersson metric and estimates (5.5), (5.6) and (5.7), we can actually show that the Ricci curvature of the perturbed Ricci metric is pinched between two negative constants. The detail will be given in our next paper.

6. Equivalent metrics on the moduli space

In this section, we prove the equivalence among the Ricci metric, perturbed Ricci metric, Kähler-Einstein metric and the McMullen metric. These equivalences imply that the Teichmüller metric is equivalent to the Kähler-Einstein metric which gives a positive answer to Yau's Conjecture. The main tool we use is the Schwarz-Yau Lemma. Also, to control the McMullen metric, we give a simple formula of the first derivative of the geodesic length functions.

Lemma 6.1. *The Weil-Petersson metric is bounded above by a constant multiple of the Ricci metric. Namely, there is a constant $\alpha > 0$ such that $\omega_{WP} \leq \alpha \omega_\tau$.*

Proof. This lemma follows from Corollary 4.1 and Corollary 4.2. It also follows directly from Schwarz-Yau Lemma.

□

By using this simple result, we have

Theorem 6.1. *The Ricci metric and the perturbed Ricci metric are equivalent.*

Proof. Since $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ and $C > 0$, we know that the Ricci metric is bounded above by the perturbed Ricci metric. By using the above lemma, we also have the bound of the other side.

□

By the work of Cheng and Yau [2] and Mok and Yau [10], there is a unique complete Kähler-Einstein metric on the moduli space whose Ricci curvature is -1. One of the main results of this section is the equivalence of the Kähler-Einstein metric and the Ricci metric. To prove this result, we need the following simple fact of linear algebra.

Lemma 6.2. *Let A and B be positive definite $n \times n$ Hermitian matrices and let α, β be positive constants such that $B \geq \alpha A$ and $\det(B) \leq \beta \det(A)$. Then there is a constant $\gamma > 0$ depending on α, β and n such that $B \leq \gamma A$.*

Theorem 6.2. *The Ricci metric is equivalent to the Kähler-Einstein metric g_{KE}.*

Proof. Consider the identity map $i : (\mathcal{M}_g, g_{KE}) \to (\mathcal{M}_g, \widetilde{\tau})$. We know that the Kähler-Einstein metric is complete and its Ricci curvature is -1. By Theorem 5.2 we know that the holomorphic sectional curvatures of the perturbed Ricci metric is bounded above by a negative constant. From the Schwarz-Yau Lemma, there is a constant $c_0 > 0$ such that

$$g_{KE} \geq c_0 \widetilde{\tau}.$$

From Theorem 6.1 we know that the Kähler-Einstein metric is bounded below by a constant multiple of the Ricci metric

$$g_{KE} \geq \tilde{c}_0 \tau. \tag{6.1}$$

Now we consider the identity map $j : (\mathcal{M}_g, \tau) \to (\mathcal{M}_g, g_{KE})$. By Theorem 4.4 we know that the Ricci curvature of the Ricci metric is bounded from below. Also, the Ricci curvature of the Kähler-Einstein metric is -1. From the Schwarz-Yau Lemma for volume forms, there is a constant $c_1 > 0$ such that

$$\det(g_{KE}) \leq c_1 \det(\tau). \tag{6.2}$$

By combining formula (6.1), (6.2) and Lemma 6.2 we have proved the theorem.

□

Now we consider the McMullen metric. In [9] McMullen constructed a new metric $g_{1/l}$ on $\mathcal{M}_g$ which is equivalent to the Teichmüller metric and is Kähler hyperbolic. More precisely, let $Log : \mathbb{R}_+ \to [0, \infty)$ be a smooth function such that

(1) $Log(x) = \log x$ if $x \geq 2$;
(2) $Log(x) = 0$ if $x \leq 1$.

For suitable choices of small constants $\delta, \epsilon > 0$, the Kähler form of the McMullen metric $g_{1/l}$ is

$$\omega_{1/l} = \omega_{WP} - i\delta \sum_{l_\gamma(X) < \epsilon} \partial\overline{\partial} Log \frac{\epsilon}{l_\gamma}$$

where the sum is taken over primitive short geodesics γ on X. We will also write this as ω_M.

To compare the Ricci metric and the McMullen metric, we compute the first order derivative of the short geodesics.

Lemma 6.3. *Let $X_0 \in \overline{\mathcal{M}}_g$ be a codimension m boundary point and let $(t_1, \cdots, s_n)$ be the pinching coordinates near X_0. Let l_j be the length of the geodesic on the collar Ω_c^j. Then*

$$\partial_i l_j = -\pi u_j \overline{b_i^j}$$

if $i \neq j$ and

$$\partial_i l_j = -\pi u_j \overline{b_i}$$

if $i = j$. Here b_i^j and b_i are defined in Lemma 4.2.

Proof. It is clear that on the genuine collar Ω_c^j, λA_i is an anti-holomorphic quadratic differential. By using the rs-coordinate z on Ω_c^j, we can denote λA_i by $\kappa_i(\overline{z}) d\overline{z}^2$. We consider the coefficient of the term $\overline{z}^{-2}$ in the expansion of κ_i and denote it by $C_{-2}(\kappa_i)$. From formula (4.2) and Lemma 4.2 we know that

$$C_{-2}(\kappa_i) = \frac{1}{2} u_j^2 \overline{b_i^j}. \tag{6.3}$$

Now we use a different way to compute $C_{-2}(\kappa_i)$. Fix (t_0, s_0) with small norm and let $X = X_{t_0, s_0}$. Let w be the rs-coordinates on the j-th collar of $X_{t,s}$ and let z be the rs-coordinate on the j-th

collar of X. Clearly $w = w(z,t,s)$ is holomorphic with respect to z and when $(t,s) = (t_0,s_0)$, we have $w = z$. We pull-back the metric on the j-th collar of $X_{t,s}$ to X. We have

$$\Lambda = \frac{1}{2}u_j^2|w|^{-2}\csc^2(u_j \log|w|)\left|\frac{\partial w}{\partial z}\right|^2$$

is the Kähler-Einstein metric on the j-th collar of $X_{t,s}$. Now from formulas (2.2) and (2.3), at point (t_0,s_0), a simple computation shows that

$$\kappa_i(\overline{z}) = -\frac{u_j\partial_i u_j}{\overline{z}^2} + \frac{u_j^2+1}{\overline{z}^3}\partial_i\overline{w}\mid_{(t_0,s_0)} - \frac{u_j^2+1}{\overline{z}^2}\partial_i\partial_{\overline{z}}\overline{w}\mid_{(t_0,s_0)} - \partial_i\partial_{\overline{z}}\partial_{\overline{z}}\partial_{\overline{z}}\overline{w}\mid_{(t_0,s_0)}. \tag{6.4}$$

From the above formula we can see that $C_{-2}(\kappa_i) = -u_j\partial_i u_j$ since the contribution of the last three terms in the above formula to $C_{-2}(\kappa_i)$ is 0. By comparing equations (6.3) and (6.4) we have

$$\partial_i u_j = -\frac{1}{2}u_j\overline{b_i^j}.$$

The lemma follows from the fact that $l_j = 2\pi u_j$. Again, the above argument also works when $i = j$. In this case, we replace b_i^j by b_i.

□

Now we can prove another main theorem of this section.

Theorem 6.3. *The Ricci metric is equivalent to the McMullen metric, the Teichmüller metric and the Kobayashi metric.*

Proof. Royden proved that the Teichmüller metric is the same as the Kobayashi metric. Also, the equivalence of the McMullen metric and the Teichmüller metric was proved by McMullen [9]. We only need to show the equivalence between the Ricci metric and the McMullen $g_{1/l}$ metric.

Since the Ricci curvature of the $g_{1/l}$ metric is bounded from below and it is complete, by the Schwarz-Yau lemma we know that

$$\tau < \widetilde{\tau} \le C_0 g_{1/l}$$

for some constant C_0. Now we prove the other bound. Fix a boundary point X_0 and the pinching coordinates near X_0. By Theorem 1.1 and Theorem 1.7 of [9] we know that there are constants c_1, c_2 such that, when $i \le m$,

$$\begin{aligned}(g_{1/l})_{i\bar{i}} = \left\|\frac{\partial}{\partial t_i}\right\|_{g_{1/l}}^2 < c_1\left\|\frac{\partial}{\partial t_i}\right\|_T^2 &\le c_2\left(\left\|\frac{\partial}{\partial t_i}\right\|_{WP}^2 + \sum_{l_\gamma<\epsilon}\left|(\partial\log l_\gamma)\frac{\partial}{\partial t_i}\right|^2\right)\\ &= c_2\left(\left\|\frac{\partial}{\partial t_i}\right\|_{WP}^2 + \sum_{j=1}^m|\partial_i\log l_j|^2\right).\end{aligned} \tag{6.5}$$

By Lemma 6.3 we know that

$$|\partial_i\log l_j|^2 = \left|\frac{-\pi u_j\overline{b_i^j}}{l_j}\right|^2 = \frac{1}{4}\left|b_i^j\right|^2.$$

From Lemma 4.2 we have

$$\sum_{j=1}^m|\partial_i\log l_j|^2 = \frac{1}{4}\frac{u_i^2}{\pi^2|t_i|^2}(1+O(u_0)).$$

From the above formulas and Corollary 4.1 and Corollary 4.2 we know that there is a constant c_3 such that

$$\left\|\frac{\partial}{\partial t_i}\right\|_{WP}^2 + \sum_{j=1}^m|\partial_i\log l_j|^2 \le c_3\tau_{i\bar{i}}$$

which implies

$$(g_{1/l})_{i\bar{i}} \le c_4 \tau_{i\bar{i}} \tag{6.6}$$

where c_4 is another constant. The same argument works when $i \ge m+1$. So formula (6.6) holds for all i. Since the McMullen metric is bounded from below by a constant multiple of the Ricci metric and the diagonal terms of its metric matrix is bounded from above by a constant multiple of the diagonal terms of matrix of the Ricci metric, a simple linear algebra fact shows that there is a constant c_5 such that

$$\tau \ge c_5 g_{1/l}.$$

The theorem follows from a compactness argument as we have used in previous sections.

□

7. Appendix: the proof of Lemma 4.10

We will prove Lemma 4.10 in this appendix which consists of some computational details. We fix a nodal surface X_0 which corresponding to a codimension m boundary point in $\mathcal{M}_g$. Let (t,s) be the pinching coordinates near X_0 such that $X_{0,0} = X_0$. Fix (t,s) with small norm, we denote $X_{t,s}$ by X. In the curvature formula (3.30), we let $i = j = k = l \le m$. The term G_2 is a summation of the following four types of terms:

(1) $I = h^{\alpha\bar{\beta}} \left\{ \sigma_1\sigma_2 \int_X \left\{ T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\bar{\beta}}) + T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}}) \right\} \, dv \right\}$ with $(\alpha,\beta) \ne (i,i)$;
(2) $II = h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}} \, dv \right\}$ with $(\alpha,\beta) \ne (i,i)$;
(3) $III = \tau^{p\bar{q}} h^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\bar{q}}) e_{\alpha\bar{\beta}} \, dv \right\} \left\{ \widetilde{\sigma}_1 \int_{X_s} \overline{\xi}_l(e_{p\bar{j}}) e_{\gamma\bar{\delta}} \, dv \right\}$
with $(p,q,\alpha,\beta,\gamma,\delta) \ne (i,i,i,i,i,i)$;
(4) $IV = \tau_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}}$ with $(p,q) \ne (i,i)$

where $T = (\Box + 1)^{-1}$. Now we check that the norm of each type is bounded by $O\big(\frac{u_i^5}{|t_i|^4}\big)$. In the following, C_0 will be a unversal constant which may change but is independent of the Riemann surface as long as (t,s) has small norm.

Case 1. We check that each term in the sum IV has the desired bound. By Corollary 4.2 and its proof we have

$$R_{i\bar{q}i\bar{i}} = \begin{cases} O\big(\frac{u_i^5}{|t_i|^3}\big), & \text{if } q \ge m+1; \\ O\big(\frac{u_i^5 u_q^3}{|t_i|^3|t_q|}\big), & \text{if } q \le m, \text{ and } q \ne i; \\ O\big(\frac{u_i^5}{|t_i|^4}\big), & \text{if } q = i. \end{cases}$$

By using the above formula and Corollary 4.1 and 4.2, and by a case by case check we have

$$|\tau_{p\bar{i}} h^{p\bar{q}} R_{i\bar{q}i\bar{i}}| = O\big(\frac{u_i^7}{|t_i|^4}\big).$$

This proves that the norm of the last term is bounded by $= O\big(\frac{u_i^5}{|t_i|^4}\big)$.

Case 2. We check that each term in the sum I has the desired bound. Firstly, when $i=j=k=l$, we have

$$\begin{aligned}
&\sigma_1\sigma_2\left\{T(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}})+T(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\right\}\\
=&2\left\{T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})+2T(\xi_i(e_{i\overline{\beta}}))\overline{\xi}_i(e_{\alpha\bar{i}})+T(\xi_i(e_{i\bar{i}}))\overline{\xi}_\beta(e_{\alpha\bar{i}})\right\}\\
&+2\left\{T(\xi_i(e_{\alpha\bar{i}}))\overline{\xi}_i(e_{i\overline{\beta}})+2T(\xi_i(e_{\alpha\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}})+T(\xi_i(e_{\alpha\bar{i}}))\overline{\xi}_\beta(e_{i\bar{i}})\right\}\\
&+2\left\{T(\xi_\alpha(e_{i\bar{i}}))\overline{\xi}_i(e_{i\overline{\beta}})+T(\xi_\alpha(e_{i\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}})+T(\xi_\alpha(e_{i\bar{i}}))\overline{\xi}_\beta(e_{i\bar{i}})\right\}\\
&+2T(\xi_\alpha(e_{i\overline{\beta}}))\overline{\xi}_i(e_{i\bar{i}}).
\end{aligned}\tag{7.1}$$

We estimate the integration of each term in the above summation. To estimate these terms, we note that, if $\alpha\neq\beta$ or $\alpha=\beta\geq m+1$, then

$$\left|h^{\alpha\overline{\beta}}\|f_{\alpha\overline{\beta}}\|_1\right|=O(1).\tag{7.2}$$

Also, we have

$$\|P(e_{\alpha\overline{\beta}})\|_0\leq\|e_{\alpha\overline{\beta}}\|_2\leq C_0\|f_{\alpha\overline{\beta}}\|_1.\tag{7.3}$$

These formulae can be checked easily by using Theorem 4.1, Corollary 4.1, Lemma 4.3 and Lemma 4.7.

Now we estimate $\left|h^{\alpha\overline{\beta}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\ dv\right|$. If $\alpha\neq\beta$ or $\alpha=\beta\geq m+1$, we have

$$\begin{aligned}
&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\ dv\right|\leq\left(\int_X|T(\xi_i(e_{i\bar{i}}))|^2\ dv\int_X|\overline{\xi}_i(e_{\alpha\overline{\beta}})|^2\ dv\right)^{\frac{1}{2}}\\
\leq&\left(\int_X|\xi_i(e_{i\bar{i}})|^2\ dv\int_X|\overline{\xi}_i(e_{\alpha\overline{\beta}})|^2\ dv\right)^{\frac{1}{2}}=\left(\int_X f_{i\bar{i}}|P(e_{i\bar{i}})|^2\ dv\int_X f_{i\bar{i}}|P(e_{\alpha\overline{\beta}})|^2\ dv\right)^{\frac{1}{2}}\\
\leq&\|P(e_{i\bar{i}})\|_0\|P(e_{\alpha\overline{\beta}})\|_0h_{i\bar{i}}\leq C_0\|f_{i\bar{i}}\|_1\|f_{\alpha\overline{\beta}}\|_1h_{i\bar{i}}=O\big(\frac{u_i^5}{|t_i|^4}\big)\|f_{\alpha\overline{\beta}}\|_1
\end{aligned}$$

since $\|f_{i\bar{i}}\|_1=O\big(\frac{u_i^2}{|t_i|^2}\big)$. Together with formula (7.2) we have

$$\left|h^{\alpha\overline{\beta}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\beta}})\ dv\right|=O\big(\frac{u_i^5}{|t_i|^4}\big).$$

If $\alpha=\beta\leq m$ and $\alpha\neq i$, we have

$$\begin{aligned}
\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\ dv\right|\leq&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\ dv\right|\\
&+\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\ dv\right|.
\end{aligned}\tag{7.4}$$

From Lemma 4.7 we have

$$\|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0\leq\|e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}}\|_2\leq\|f_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}}\|_1=O\big(\frac{u_\alpha^4}{|t_\alpha|^2}\big).$$

So

$$
\begin{aligned}
&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\ dv\right| \le \|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left|\int_X |T(\xi_i(e_{i\bar{i}}))||A_i|\ dv\right| \\
\le&\|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X |T(\xi_i(e_{i\bar{i}}))|^2\ dv \int_X f_{i\bar{i}}\ dv\right)^{\frac{1}{2}} \\
\le&\|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X |\xi_i(e_{i\bar{i}})|^2\ dv \int_X f_{i\bar{i}}\ dv\right)^{\frac{1}{2}} \qquad (7.5)\\
=&\|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0 \left(\int_X f_{i\bar{i}}|P(e_{i\bar{i}})|^2\ dv \int_X f_{i\bar{i}}\ dv\right)^{\frac{1}{2}} \\
\le&\|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0\|e_{i\bar{i}}\|_2 h_{i\bar{i}} = O(\frac{u_\alpha^4}{|t_\alpha|^2})O(\frac{u_i^5}{|t_i|^4}).
\end{aligned}
$$

Since the support of $\widetilde{e_{\alpha\overline{\alpha}}}$ is inside Ω_c^α, we know the support of $P(\widetilde{e_{\alpha\overline{\alpha}}})$ is inside Ω_c^α. From Lemma 4.8 we have

$$
\begin{aligned}
&\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\ dv\right| = \left|\int_{\Omega_c^\alpha} T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(\widetilde{e_{\alpha\overline{\alpha}}})\ dv\right| \\
\le&\|A_i\|_{0,\Omega_c^\alpha}\|T(\xi_i(e_{i\bar{i}}))\|_0|P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} \le \|A_i\|_{0,\Omega_c^\alpha}\|\xi_i(e_{i\bar{i}})\|_0|P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} \qquad (7.6)\\
=&\|A_i\|_{0,\Omega_c^\alpha}\|A_i\|_0\|P(e_{i\bar{i}})\|_0|P(\widetilde{e_{\alpha\overline{\alpha}}})|_{L^1} = O(\frac{u_i^3}{|t_i|})O(\frac{u_i}{|t_i|})O(\frac{u_i^2}{|t_i|^2})O(\frac{u_\alpha^3}{|t_\alpha|^2}) \\
=&O(\frac{u_i^6}{|t_i|^4})O(\frac{u_\alpha^3}{|t_\alpha|^2}).
\end{aligned}
$$

By combining the inequalities (7.5) and (7.6) we know that

$$
\left|\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\ dv\right| = O(\frac{u_i^5}{|t_i|^4})O(\frac{u_\alpha^3}{|t_\alpha|^2}).
$$

From Lemma 4.1 we have

$$
\left|h^{\alpha\overline{\alpha}}\int_X T(\xi_i(e_{i\bar{i}}))\overline{\xi}_i(e_{\alpha\overline{\alpha}})\ dv\right| = O(\frac{u_i^5}{|t_i|^4}).
$$

We finish the estimate of the first term in the sum (7.1). The integration of other terms in this sum can be estimated in a similar way.

Case 3. We check that each term in the sum III has the desired bound. By Lemma 4.2 we first prove that when $q \neq i$ and $k = i$,

$$
\left|h^{\alpha\overline{\beta}}\left\{\sigma_1\int_X \xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\ dv\right\}\right| = \begin{cases} O(\frac{u_i^{\frac{5}{2}}}{|t_i|^2})O(\frac{u_q}{|t_q|}) \text{ if } q \le m \\ O(\frac{u_i^{\frac{5}{2}}}{|t_i|^2}) \text{ if } q \ge m+1 \end{cases} \qquad (7.7)
$$

Again, we do a case by base check. First we estimate $\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right|$. If $\alpha\neq\beta$ or $\alpha=\beta\geq m+1$, we have

$$\begin{aligned}
&\left|\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right| = \left|\int_X e_{i\overline{q}}\xi_i(e_{\alpha\overline{\beta}})\,dv\right| \leq \left(\int_X |\xi_i(e_{\alpha\overline{\beta}})|^2\,dv\int_X |e_{i\overline{q}}|^2\,dv\right)^{\frac{1}{2}}\\
&\leq \left(\int_X f_{i\overline{i}}|P(e_{\alpha\overline{\beta}})|^2\,dv\int_X |f_{i\overline{q}}|^2\,dv\right)^{\frac{1}{2}} \leq \|P(e_{\alpha\overline{\beta}})\|_0\left(\int_X f_{i\overline{i}}\,dv\int_X f_{i\overline{i}}f_{q\overline{q}}\,dv\right)^{\frac{1}{2}}\\
&\leq \|P(e_{\alpha\overline{\beta}})\|_0\|A_q\|_0 h_{i\overline{i}} = O\big(\frac{u_i^3}{|t_i|^2}\big)\|f_{\alpha\overline{\beta}}\|_1\|A_q\|_0.
\end{aligned}\tag{7.8}$$

This implies

$$\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right| = O\big(\frac{u_i^3}{|t_i|^2}\big)\|A_q\|_0.$$

If $\alpha=\beta\leq m$ and $\alpha\neq i$, we have

$$\left|\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\alpha}}\,dv\right| \leq \left|\int_X \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\,dv\right| + \left|\int_X \xi_i(e_{i\overline{q}})(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right|.$$

For the second term in the above formula, we have

$$\begin{aligned}
&\left|\int_X \xi_i(e_{i\overline{q}})(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right| = \left|\int_X e_{i\overline{q}}\xi_i(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\,dv\right|\\
&\leq \left(\int_X |e_{i\overline{q}}|^2\,dv\int_X |\xi_i(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})|^2\,dv\right)^{\frac{1}{2}} \leq \left(\int_X |f_{i\overline{q}}|^2\,dv\int_X f_{i\overline{i}}|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})|^2\,dv\right)^{\frac{1}{2}}\\
&\leq \|P(e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}})\|_0\left(\int_X f_{i\overline{i}}f_{q\overline{q}}\,dv\int_X f_{i\overline{i}}\,dv\right)^{\frac{1}{2}} \leq \|e_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}}\|_2\|A_q\|_0 h_{i\overline{i}}\\
&\leq \|f_{\alpha\overline{\alpha}}-\widetilde{e_{\alpha\overline{\alpha}}}\|_2\|A_q\|_0 h_{i\overline{i}} = O\big(\frac{u_\alpha^4}{|t_\alpha|^2}\big)O\big(\frac{u_i^3}{|t_i|^2}\big)\|A_q\|_0.
\end{aligned}$$

For the first term in the above formula, we have

$$\begin{aligned}
&\left|\int_X \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\,dv\right| = \left|\int_{\Omega_c^\alpha} \xi_i(e_{i\overline{q}})\widetilde{e_{\alpha\overline{\alpha}}}\,dv\right| \leq \|A_i\|_{0,\Omega_c^\alpha}\|P(e_{i\overline{q}})\|_0\int_{\Omega_c^\alpha}\widetilde{e_{\alpha\overline{\alpha}}}\,dv\\
&\leq \|A_i\|_{0,\Omega_c^\alpha}\|e_{i\overline{q}}\|_2\int_{\Omega_c^\alpha}\widetilde{e_{\alpha\overline{\alpha}}}\,dv \leq O\big(\frac{u_\alpha^3}{|t_\alpha|^2}\big)O\big(\frac{u_i^3}{|t_i|}\big)\|f_{i\overline{q}}\|_1.
\end{aligned}$$

By combining the above two formulas we have the desired bound for $\left|h^{\alpha\overline{\alpha}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\alpha}}\,dv\right|$.

When $\alpha=\beta=i$, by using a similar method we can show that $\left|h^{i\overline{i}}\int_X \xi_i(e_{i\overline{q}})e_{i\overline{i}}\,dv\right| = O\big(\frac{u_i^3}{|t_i|^2}\big)\|A_q\|_0$. From the above estimates we have proved that the term $\left|h^{\alpha\overline{\beta}}\int_X \xi_i(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right|$ in formula (7.7) has the desired estimate. By using similar method we can show that the other terms in (7.7) have the desired estimate. This proves formula (7.7).

In a similar way, in the case $q=i$ we can prove that, when $k=i$,

$$\left|h^{\alpha\overline{\beta}}\left\{\sigma_1\int_X \xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\right\}\right| = \begin{cases} O\big(\frac{u_i^3}{|t_i|^3}\big), & \text{if } \alpha=\beta=i;\\ O\big(\frac{u_i^4}{|t_i|^3}\big), & \text{if } \alpha\neq i \text{ or } \beta\neq i.\end{cases}\tag{7.9}$$

By combining formulas (7.8) and (7.9) we conclude that each term in the sum III is of order $O\big(\frac{u_i^5}{|t_i|^4}\big)$.

Case 4. We need to show that each term in the sum II is of order $O\left(\frac{u_i^5}{|t_i|^4}\right)$. This case can be proved by a case by case check by using the similar estimates as in the third case together with Lemma 4.9. This finishes the proof.

□

Remark 7.1. The method we estimate these terms can be directly applied to the computations of the full curvature tensor and we can get certain bounds for the bisectional curvature and the Ricci curvature of the Ricci metric as well as the perturbed Ricci metric.

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